Two element subsets

Counting

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Selection sort

Let's return to the selection sort. Again asking for the number of comparisons made in Line 3.

Input Array `A` of length `n`

Output The same array `A` sorted in place.

```
1  for i := 1 to n − 1
2      for j := i + 1 to n
3          if A_i > A_j
4              swap A_i and A_j
```

Question *Can we solve this problem using the product principle?*

Two-element subset

- Every possible pair A_i, A_j is compared exactly once
- Line 3 is run for every choice of (i, j) where $1 \leq i < j \leq n$.
- In other words, we need to count two-element subsets

$$\{i, j\} = \{1, 2, \ldots, n\}$$

Selecting an ordered pair

Let's first count the number of ordered pairs (i, j) where $i \neq j$.

1. Choose $i \in \{1, 2, \ldots, n\}$
   - You have n choices.
2. Given i, choose $j \in S_i$ where

$$S_i = \{1, 2, \ldots, i-1, i+1, i+2, \ldots, n\} = \{1, 2, \ldots, n\} \backslash \{i\}$$

   - You have $|S_i| = n - 1$ options (i is no longer available)
3. The total set of eligible ordered pairs is

$$S = \bigcup_{i=1}^{n} \{(i, j) \mid j \in S_i\}$$

4. What is $|S|$?

The Product Principle

- We have a partitioning

$$S = \bigcup_{i=1}^{n} S_i, \quad \text{and } |S_i| = n - 1.$$

- What is $|S|$?
- The product principle applies,
  - to n partitions of size $n - 1$ each.
- $|S| = n(n - 1)$

Selecting an unordered pair

There are $|S| = n(n-1)$ *ordered pairs in the set* $\{1, 2, \ldots, n\}$.

- We only want unordered pairs
  - In terms of sets $\{i, j\} = \{j, i\}$
  - Alternatively we accept (i, j) where $i < j$ and reject (j, i).
- Either way, each pair appears twice in S
  - we divide by two
- The number of unordered pairs is

$$\binom{n}{2} = \frac{n(n-1)}{2} = \sum_{i=1}^{n} i$$

(Dividing by two we also used the Quotient Principle. We will return to that later.)

Exercise

Five mates want to arrange a private tournament of table tennis. Each participant should play each of the others exactly ones.

1. How many matches are required?
2. How does this relate to counting of two-element subsets?

Summary

Counting objects is an important aspect of mathematics, science, and engineering.

- Abstraction of concrete objects into abstract sets.
- Partitioning a set into blocks
- Sum Principle
- Product Principle
- Ordered and unordered pairs