Bijections

Lists, permutations, and subsets

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Classes of functions

Recall

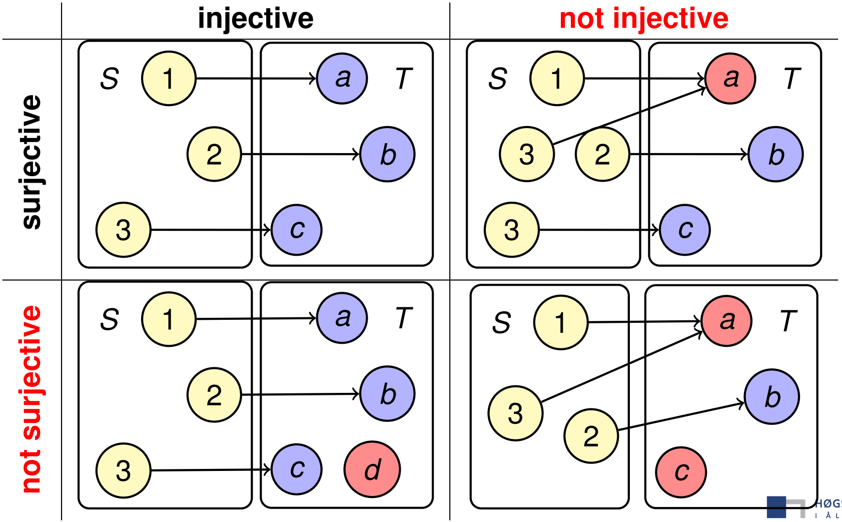
$$f : \{1, 2\} \to \{a, b\}, \tag{1}$$
$$f_1(1) = a \quad \text{and} \quad f_1(2) = a \tag{2}$$
$$f_2(1) = a \quad \text{and} \quad f_2(2) = b \tag{3}$$
$$f_3(1) = b \quad \text{and} \quad f_3(2) = a \tag{4}$$
$$f_4(1) = b \quad \text{and} \quad f_4(2) = b \tag{5}$$

Function as a look-up table

S	T			
	$f_1(S)$	$f_2(S)$	$f_3(S)$	$f_4(S)$
1	a	a	b	b
2	a	b	a	b

- f_2 and f_3 are onto or surjective,
  - every element of the codomain T is used
  - for any $y \in T$, we can find $x \in S$ so that $f(x) = y$
- f_2 and f_3 are also injective,
  - every element $f(x)$ is different
  - we cannot find $x \neq y$ in S so that $f(x) = f(y)$

Function classes



Bijections

Definition (Bijection)

A function f which is both surjective and injective is called a bijection.

- Bijections are important
- one-to-one relationship between sets
- One set serves as a representation of the other

Definition (Bijection Principle)

If there is a bijection $f : S \to T$ then S and T have the same number of elements; i.e. $|S| = |T|$.

Another sample of code

This algorithm counts triangles in an array `A` *of* `n` *points.*

```
1 trianglecount := 0
2 for i := 1 to n
3     for j := i+1 to n
4         for k := j+1 to n
5             if A_i, A_j, A_k are not collinear
6                 increment trianglecount
```

How many times is the collinearity check (Line 4) run?

First bijection

```
for i := 1 to n
    for j := i+1 to n
        for k := j+1 to n
```

- Loop for every triple (i, j, k) where $0 < i < j < k \leq n$.
- Bijection $f : P \to S$, where
  - P is the set of iterations
  - S is the set of increasingly ordered triples (i, j, k) from $\mathbb{N}_n$.

Second bijection

- Let T be the set of three-element subsets of $\mathbb{N}_n$
- A triple $(i, j, k) \in S$ where $i < j < k$ corresponds to
  - a set $\{i, j, k\} \in T$
  - why?
- Since i, j, k are distinct, there is an obvious map $g : (i, j, k) \mapsto \{i, j, k\}$

surjective: for any $\{i, j, k\} \in T$ we can put i, j, k in increasing order to form a triple x, and $g(x) = \{i, j, k\}$.

injective: only one ordering of i, j, k gives an element of S, making a unique x such that $g(x) = \{i, j, k\}$.

$$|P| = |S| = |T|$$

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Exercise

- Consider weekend activities
  - Set of activities $A = \{\text{Horseriding}, \text{Badminton}, \text{BBQ}\}$
  - Set of days $D = \{\text{Saturday}, \text{Sunday}\}$

1. List all possible functions $A \to D$
2. List all possible functions $D \to A$
3. Which of the functions are injective?
4. Which of the functions are surjective?