The Bijection Principle

Using sets of equal size

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Bijections

Definition (Bijection)

A function f which is both surjective and injective is called a bijection.

- Bijections are important
- one-to-one relationship between sets
- One set serves as a representation of the other

Definition (Bijection Principle)

If there is a bijection $f : S \to T$ then S and T have the same number of elements; i.e. $|S| = |T|$.

Another sample of code

This algorithm counts triangles in an array `A` *of* `n` *points.*

```
1 trianglecount := 0
2 for i := 1 to n
3     for j := i+1 to n
4         for k := j+1 to n
5             if A_i, A_j, A_k are not collinear
6                 increment trianglecount
```

How many times is the collinearity check (Line 5) run?

First bijection

```
for i := 1 to n
    for j := i+1 to n
        for k := j+1 to n
```

- Loop for every triple (i, j, k) where $0 < i < j < k \leq n$.
- Bijection $f : P \to S$, where
  - P is the set of iterations
  - S is the set of increasingly ordered triples (i, j, k) from $\mathbb{N}_n$.

Second bijection

- Let T be the set of three-element subsets of $\mathbb{N}_n$
- A triple $(i, j, k) \in S$ where $i < j < k$ corresponds to
  - a set $\{i, j, k\} \in T$
  - why?
- Since i, j, k are distinct, there is an obvious map $g : (i, j, k) \mapsto \{i, j, k\}$

surjective: for any $\{i, j, k\} \in T$ we can put i, j, k in increasing order to form a triple x, and $g(x) = \{i, j, k\}$.

injective: only one ordering of i, j, k gives an element of S, making a unique x such that $g(x) = \{i, j, k\}$.

$$|P| = |S| = |T| = \binom{n}{3}$$

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Exercise

Consider selection sort as studied in previous videos, where we wanted to count the number of executions of line 3 (the comparison).

Now we want to use the bijection principle to map this counting problem into a more generic counting problem.

```
1  for  i = 1, 2, ..., n−1
2      for  j = i+1, i+2, ..., n
3          if  A_i > A_j
4              swap  A_i  with  A_j
```

Hint! You can use the method for counting subsets. How do the loop indices (i, j) *relate to subsets?*