The *less than* relation

Partial and total orders

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The *less than* relation

- The most well-known and most used relations

$$<$$

- Many relations have similar properties

Ordering

- $<$ defines a sort order
- If $<$ is a relation on a set S we can define sorted lists from S.

Definition

A list $[x_1, \ldots, x_n]$ is sorted if and only if $x_i < x_j$ implies that $i < j$.

- Obviously $>$ has exactly the same properties.
- and $\geq, \leq$ have similar properties.

We will look at other relations which provide an ordering.

Are there other *smaller than* relations?

1. neighbour of
2. set equivalence
3. subset of

Are any of these smaller than relations?

Symmetry and antisymmetry

- View the sorted list
  - $[x_1, x_2, x_3, \ldots, x_n]$
- *neighbour of* is symmetric
  - i.e. if $x_i \sim x_j$ then $x_j \sim x_i$
  - x_i should be both before and after x_j
- *set equivalence* is alse symmetric
  - same problem
- *subset of* is not symmetric
  - it is in fact anti-symmetric
  - if $x_i \subset x_j$ then $x_j \not\subset x_i$ (except if $x_i = x_j$)

Definition

Anti-symmetry

Definition

A relation R is anti-symmetric if xRy and yRx implies that $x = y$.

Transitivity

- View again the sorted list
  - $[x_1, x_2, x_3, \ldots, x_n]$
- Suppose $x_i < x_j$ and $x_j < x_k$ $(i < j < k)$
  - what would you say about x_i in relation to x_k?
- We cannot have $x_k < x_i$, lest the list be unsorted
- We would expect that $x_i < x_k$, i.e. transitivity

The subset relation $\subset$ is transitive.

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Transitivity

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To be or not to be ... equal

- *Smaller than* relations come in two variants
  - Non-reflexive: $<$
  - Reflexive: $\leq$
- Question is, do you include (x, x) in the relation?
- Same for the subset relation
  - $\subsetneq$ versus $\subset$
  - or $\subset$ versus $\subseteq$
- Each is well-defined in terms of the other

Partial ordered set

Definition (Partial order)

A partial order is a relation $\prec$ which is reflexive, transitive, and anti-symmetric.

- Note that elements may be incomparable
  - Neither $x \prec y$ nor $y \prec x$
- A partial order defines a sort order
  - but the sorted list may not be unique

Definition

A partially ordered set (or poset) S is a set with some partial order $\prec$.

Totally ordered set

Definition

A total order is a partial order $\prec$ where either $x \prec y$ or $y \prec x$ for any pair (x, y).

- That is, every pair of elements is comparable
- A total order defines a unique sort order for any set

Definition

A totally ordered set S is a set with some total order $\prec$.

Exercise

1. Consider the set of people, and the relation *is an ancestor of*, where a person is considered to be one of his own ancestors.
   - Is this relation a partial order?
   - Is it a total order?
2. What about the relation *is a parent of*?
3. Give reasons for your answers.