The symmetry principle

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Recollection

Remember the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We defined this as the number of k-element subsets of an n-element set.

- Last talk, we asked about the relationship between

$$\binom{n}{k} \quad \text{and} \quad \binom{n}{n-k}$$

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Symmetry of the Binomial Coefficient

- Having solved the previous exercise, we know that $\binom{12}{3} = \binom{12}{9} = 220$
- In general, we may suspect $\binom{n}{k} = \binom{n}{n-k}$
- How do we prove it?
- Two approaches
  - Abstract, symbolic manipulation
  - Intuition

Symmetry of the Binomial Coefficient

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The abstract, symbolic approach

... without knowing what n and k means ...

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Inserting for $k = n - k$, we get

$$\binom{n}{n-k} = \frac{n!}{(n-k)!k!}$$

- Since the order of the factors is irrelevant, we conclude that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

- Symmetry in k and $n - k$

The symmetry principle

If a mathematical formula has a symmetry, e.g. two variable may be swapped, then a proof explaining the symmetry will usually add insight.

The intutive way

- Choosing a k-set $S \subset T$ for some n-set T,
  - We choose in fact two sets

1. The k-set S of elements we want
2. The $(n-k)$-set $T \backslash S$ of elements we throw away

- One-to-one mapping between k- and $(n-k)$-sets $T \subset S$
- Hence $\binom{n}{k} = \binom{n}{n-k}$

Conclusion

Intuition confirms the theory.

- The symmetry is fairly easy to justify.

  In many cases, symmetry can be used to simplify formulæ and arguments.