Simplification, rewriting, and proof

Predicate logic

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Example

$$\exists x \in \mathbb{R}^+, (x > 1), \tag{1}$$

$$\forall x \in \mathbb{R}^+, (x > 1), \tag{2}$$

Rewriting with the existential quantifier

It may be useful to get rid of $\mathbb{R}^+$, using instead a more universal set, such as $\mathbb{R}$.

$$\exists x \in \mathbb{R}^+ \ (x > 1)$$

- Two properties,
  1. $x \in \mathbb{R}^+ = \{a \mid a \in \mathbb{R} \land a \geq 0\}$
  2. $x > 1$
- $\exists x, \ x \in \mathbb{R}^+ \land x > 1$

$$\exists x \in \mathbb{R} \ (x \geq 0 \land x > 1)$$

Rewriting with the universal quantifier

$$\forall x \in \mathbb{R}^+ \ (x > 1)$$

- Same properties,
  1. $x \in \mathbb{R}^+ = \{a \mid a \in \mathbb{R} \wedge a \geq 0\}$
  2. $x > 1$
- Universal quantifier means that
  - Property 2 holds whenever Property 1 holds

$$\forall x \in \mathbb{R} \ (x \geq 0 \Rightarrow x > 1)$$

Theorem

- Universes U_2 and $U_1 \subset U_2$
- Statement $q(x)$ (over U_2)
- Suppose $U_1 = \{x \mid x \in U_2 \wedge q(x)\}$
- Another statement $p(x)$ over U_2

$$\forall x \in U_1, p(x) \text{ equivalent to } \forall x \in U_2, (q(x) \Rightarrow p(x))$$
$$\exists x \in U_1, p(x) \text{ equivalent to } \exists x \in U_2, (q(x) \wedge p(x))$$

Proof for the universal quantifier

One way implication

$$\forall x \in U_1, p(x) \text{ implies } \forall x \in U_2, (q(x) \Rightarrow p(x))$$

- Consider $x \in U_2$
- We need to prove $q(x) \Rightarrow p(x)$
- If $q(x)$ then $x \in U_1$ (by definition of U_1)
- Hence $p(x)$ by the left hand side

Proof for the universal quantifier

The other way

$$\forall x \in U_2, (q(x) \Rightarrow p(x)) \text{ implies } \forall x \in U_1, p(x)$$

- Consider $x \in U_1$
- We need to prove $p(x)$
- $q(x)$ (by the definition of U_1)
- Now $x \in U_2$ (by definition of U_1)
- Hence $q(x) \Rightarrow p(x)$ (by left hand side)
- Hence $p(x)$, quod erat demonstrandum

Proof for the existential quantifier

One way implication

$$\exists x \in U_1, p(x) \text{ implies } \exists x \in U_2, (q(x) \wedge p(x))$$

- Suppose $x \in U_1$ and $p(x)$
- Then $x \in U_2$ (because $x \in U_1$)
- $q(x)$ (because $x \in U_1$)
- $q(x) \wedge p(x)$ (both have been proven)

Proof for the existential quantifier

The other way

$$\exists x \in U_2, (q(x) \wedge p(x)) \text{ implies } \exists x \in U_1, p(x)$$

- Consider $x \in U_2$ such that $q(x) \wedge p(x)$
- We need to prove $x \in U_1$ and $p(x)$
- $x \in U_1$ (because $q(x)$)
- ... and $p(x)$ holds,

Exercise

Consider the following slight modification of the Theorem:

$$s_1 := \forall x \in U_1, p(x) \text{ equivalent to } \forall x \in U_2, (q(x) \Rightarrow p(x))$$
$$s_2 := \exists x \in U_1, p(x) \text{ equivalent to } \exists x \in U_2, (q(x) \wedge p(x))$$

For each statement s_1 and s_2, argue that it is true, or give a counter-example.

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