# Week 1: Counting

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12th November 2015

**Period** 19-25 August 2015

**Reading** This part roughly follows Chapter 1 in Stein et al's textbook. References to Rosen's textbook have been added, but some of the concepts and terms used by Stein et al and myself will be missing from Rosen.

**Programme** This document details the programme for the week, including exercises and pointers to other material. It is available in two versions:

1. as a PDF document.
2. as a web site. This depends on MathML and may require firefox/iceweasel to display correctly.

The web version includes inline video. The pdf version shows a still image from the video, providing a hyperlink directly to the video on the web site.

## 1 Session 1

In this session we will focus on practical problems, where some are given as worked examples and others at practice problems. It is important that you study the worked examples on video prior to class. The problems are paired, so that a worked example is followed by a very similar practice problem. We will discuss the solution of the practice problems in class, but please think through how you want to solve the problem in advance.

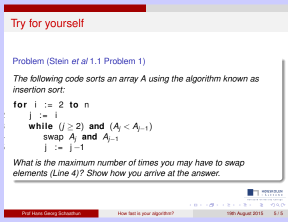
**Related reading:** Stein *et al.* p. 31–36 or Rosen p. 375–382

### 1.1 Algorithms

Worked Example

**Problem 1.1** *The following code sorts an array $A$ using the algorithm known as selection sort:*

```
1 for i := 1 to n−1
2     for j := i+1 to n
3         if A[i] > A[j]
4             swap A[i] with A[j]
```

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*What is the maximum number of times you may have to swap elements (Line 4)? Show how you arrive at the answer.*

**Problem 1.2 (Stein *et al* 1.1 Problem 1)** *The following code sorts an array $A$ using the algorithm known as insertion sort:*

```
1 for i := 2 to n
2     j := i
3     while (j ≥ 2) and (A_j < A_{j−1})
4         swap A_j and A_{j−1}
5         j := j−1
```

*What is the maximum number of times you may have to swap elements (Line 4)? Show how you arrive at the answer.*

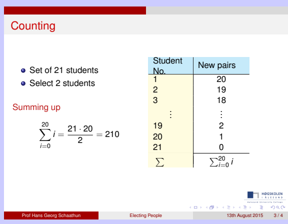
**Problem 1.3** *In Problems 1.1 and 1.2 the number of swaps may vary between lists. It is useful to describe the* worst case*, i.e. the lists which require the largest number of swaps.*

1. *What is the worst case for selection sort?*
2. *What is the worst case for insertion sort?*

## 1.2 Electing people

Worked Example

**Problem 1.4** *A class of 21 students has to elect two student rep's. How many different choices do they have for two students out of 21?*

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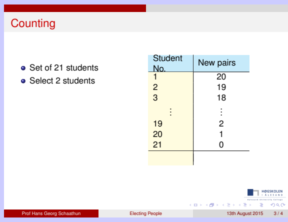
**Problem 1.5** *A club of 12 people will elect two members for a steering committee. How many different choices do they have for two people out of 12?*

**Worked Example**

**Problem 1.6** *A class of 21 students has to elect one student rep and one deputy (who cannot be the same as the main rep). How many different choices do they have for a rep and a deputy?*

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Slides

**Problem 1.7** *A club of 12 people will elect a president and a treasurer. How many different choices do they have for these two roles when the same person cannot hold both posts?*

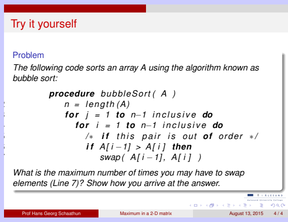
## 1.3 More algorithms

**Worked Example**

**Problem 1.8** *The following algorithm finds the largest element in a 2-D array A.*

```
1  c := A[1,1]
2  for i := 1 to n
3      for j := 1 to m
4          c := max(c, A[i,j])
5  return c
```

How many calls to `max` (line 4) are required?

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**Problem 1.9** *The following code sorts an array A using the algorithm known as bubble sort:*

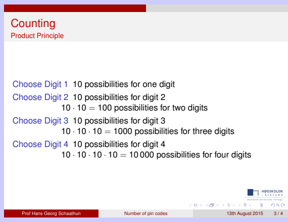
```
1   procedure bubbleSort( A )
2     n = length(A)
3     for j = 1 to n-1 inclusive do
4       for i = 1 to n-1 inclusive do
5         /* if this pair is out of order */
6         if A[i-1] > A[i] then
7           swap( A[i-1], A[i] )
8         end if
9       end for
10    end for
11  end procedure
```

*What is the maximum number of times you may have to swap elements (Line 7)? Show how you arrive at the answer.*

## 1.4 PINs and passwords

**WORKED EXAMPLE**

**Problem 1.10** *A four-digit PIN code is commonly used to authenticate users of bank cards, computer systems, key cards, etc. To make a secure system, it is important that the number of possible PINs (or passwords) is large. How many different values of the PIN code exist?*

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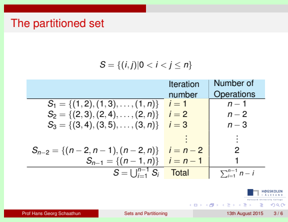
**Problem 1.11** *Imagine a computer system which uses passwords composed of exactly three Latin letters (case insensitive a to z). How many different passwords exist?*

## 1.5 Some theory

**THEORY**

In the examples and exercises we have used only one or two techniques over and over again. To be able to identify these techniques, and apply them quickly and correctly to new problems, we need some mathematical theory.

This video casts the counting problem in terms of set theory, and introduce *partitioning* and the *sum principle*.

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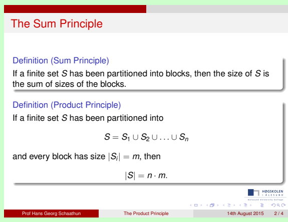
**THEORY**

Using the sum principle, it sometimes happen that all the partitions have the same size. Then the sum principle turns into a product principle. Additionally, the product principle also cover a range of special cases which are easily solved when you know how.

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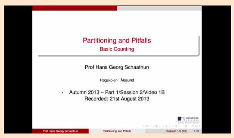
**SUPPORTING THEORY**

When partitioning a set, there are certain pitfalls to avoid. In particular, if the partitions are not disjoint, you end up double-counting. Watch the video to learn more.

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## 1.6 Sports tournaments

**Problem 1.12** *Five mates want to arrange a private tournament of table tennis. Each participant should play each of the others exactly once. How many matches are required?*

**Problem 1.13** *Five mates want to arrange a private tournament of table tennis. Each participant should play each of the others exactly once on his own home table (and once on the opponent's home table). How many matches are required? Answer this question in two ways.*

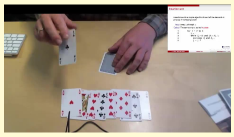
1. *Give a simple reasoning using the result from Exercise 1.12.*
2. *Forget the previous answers, and explain how you can answer the question using the product principle (Problems 1.6 and 1.7).*

# 2 Optional Background Lectures

**Background Material**

We have counted the number of swaps required in two different sorting algorithms. We could do that without any understanding of how the the algorithm works.

If you still want to understand, these two videos demonstrate the sorting algorithms.



**Background Material**

In several counting problems, we have used a Sum Principle to arrive at something like $\sum_{i=1}^{n} i$. We have tacitly used Gauss' Summation formula. The video gives the reasoning behind it.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**Theorem 1 (Gauss' sumation formula)**

$$\sum_{i=1}^{n} i = \frac{(n+1)\cdot n}{2}.$$

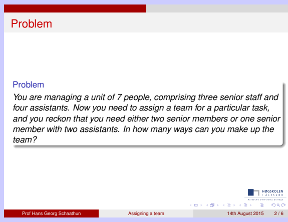
**Related reading:** Stein *et al* p. 33-34 or Rosen p. 241 (links).

# 3 Session 2

**Related reading:** Stein *et al.* p. 42–48 or Rosen p. 396–400

WORKED EXAMPLE

**Problem 3.1** *You are managing a unit of 7 people, comprising three senior staff and four assistants. Now you need to assign a team for a particular task, and you reckon that you need either two senior members or one senior member with two assistants. In how many ways can you make up the team?*

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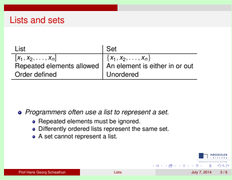
**Problem 3.2** *How many valid phone numbers exist? This question is non-trivial, since not all phone numbers have the same length. Typically, a prefix determines the length of the number. Let's consider a number plan for some imaginary country.*

*A valid number is either a three-digit number starting with 1, or a seven-digit number starting with 2–8. (Numbers starting with 0 or 9 are not yet used.) How valid numbers exist?*

THEORY

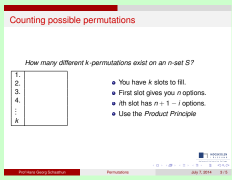
Passwords (Exercise 1.10 and 1.11) is an example of *lists*. Many other counting problems also correspond to lists, so let's explore this concept in further detail in this video.




Slides

THEORY

Besides lists, *k-permuations* is another mathematical concept which can be used to describe many counting problems. This is defined in this video.




Slides

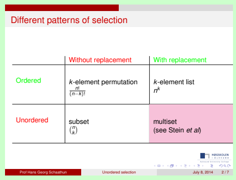
THEORY

This very short video explores the *full permutation*, a $k$-element permutation on a set of $k$ elements.


Slides

**THEORY**

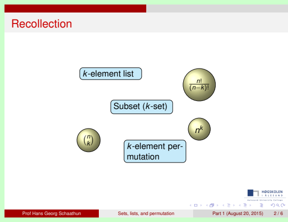
The third type of combination which commonly occur in counting problems is a subset. This video explains how to count the number of subsets of a given set.




Slides

**THEORY**

We have introduced *lists*, *k-permutations*, and subsets, each with their own counting formulæ. This video summarises the different concepts and highlights the differences.

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**Problem 3.3** *In Problems 1.12 and 1.13, we considered two different counting problems, namely the number of matches needed for a tournament of five players when*

1. *each player meets every other once and once only, and*
2. *each player meets every other once on his home table and once away.*

*For each of these problems, explain how we can use either lists, subsets, or $k$-permutations to solve the problem.*

**Problem 3.4** *You are going to appoint bishops for three dioceses. There are five candidates, and it is possible for bishop to hold multiple dioceses. In how many different ways can the appointments be made? Give reasons for the answer.*

**Problem 3.5**

1. *List all the permutations of $\{1, 2, 3\}$.*
2. *List all the two-element permutations of $\{1, 2, 3\}$.*

**Problem 3.6** *You are going to hand out $k$ distinctly coloured balloons at a birthday party of $n$ children.*

1. *In how many ways can the $k$ balloons be distributed to the $n$ children, with no limit on the number of balloons a single child may receive?*
2. *Suppose you may give at most one balloon per child. If $k \leq n$, how many combinations are possible?*
3. *Again, allowing at most one balloon per child, but $k > n$. How many combinations are possible?*

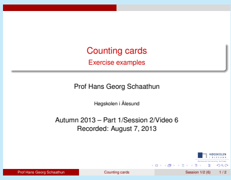
*Give reasons for your answers?*

**Problem 3.7** *Calculate the following*

1. $\binom{3}{1}$
2. $\binom{5}{2}$
3. $\binom{12}{3}$
4. $\binom{12}{9}$.
5. $\binom{712}{1}$
6. $\binom{712}{710}$

# 4 Optional problems with solution

**EXERCISE EXAMPLE**




**Problem 4.1** *In how many ways can you*

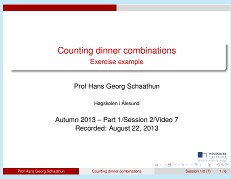
1. *draw a first card and then a second card from a 52-card deck?*
2. *draw a two cards from a 52-card deck?*
3. *draw a first, second, and third card from a 52-card deck?*

*Give reasons for your answers?*

**EXERCISE EXAMPLE**

**Problem 4.2 (Video Solution «Counting dinners»)** *A dinner meal ought to comprise both starch and protein. Suppose you have the options of potatoes, rice, and spaghetti for the starch and beef, chicken, or meatballs for the protein.*

*How many different dinners can you cook? Assume that you are allowed only one ingredient of each type.*

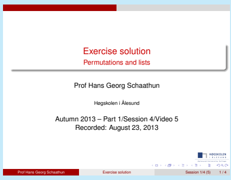



Slides
Solution

**EXERCISE EXAMPLE**

**Problem 4.3** *You take part in an urban orienteering race, where you have to visit three out of five posts in any order. The posts are, say, Fjellstuen, the church at Aspøya, Kremmergården, Gågaten, and Byparken.*

*How many iteneraries are possible.*

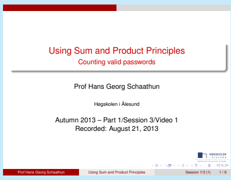



Slides
Solution

## Exercise Example

**Problem 4.4 (Paraphrased from Stein et al. Section 1.2)** *A password (for some computer system) is between four and eight characters long (inclusive), and composed of lowercase and/or uppercase letters (26-letter alphabet).*

1. *How many passwords are possible?*
2. *What counting principle(s) do you use?*
3. *What percentage of valid passwords have exactly four letters?*




Slides

## Exercise Example

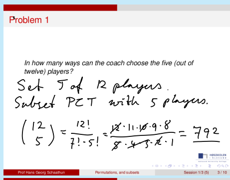
**Problem 4.5** *A basketball team has twelve players, with five players on the field at any time.*

**1** *In how many ways can the coach choose the five players?*

*To be more realistic, the five players are usually two guards, one center, and two forwards.*

**2** *In how many ways can the coach choose the $2+1+2$ players, given that he has five guards, four forwards, and thre centres on the team?*

**3** *Suppose one of the three centres is also skilled as a forward. How many ways are there to choose the five players now?*

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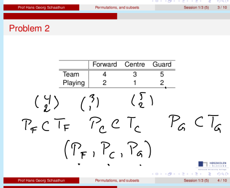


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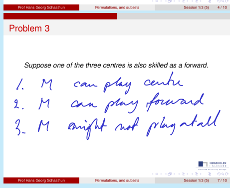


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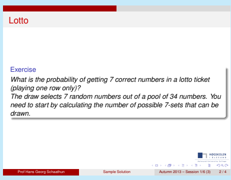
Solutions — Slide notes

## Exercise Example

**Problem 4.6 (Video Solution «Lotto»)** *What is the probability of getting 7 correct numbers in a lotto ticket (playing one row only)?*

*The draw selects 7 random numbers out of a pool of 34 numbers. You need to start by calculating the number of possible 7-sets that can be drawn.*




Slides
Solution

# 5 Last lecture topics (video)

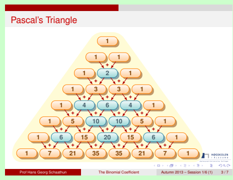
The following video lectures elaborate on some of topics we have considered. It is recommended that you watch them before the end of the week.

**THEORY**

The name *binomial coefficient* is an interesting object.

**Definition 1** *The binomial coefficient,* $n$ choose $k$ *is written*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

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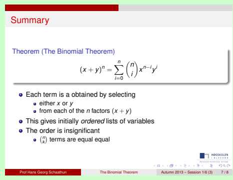
**Related reading:** Stein *et al.* p. 52–56 or Rosen p. 406–407

**THEORY**

The name *binomial coefficient* stems from a particular application, namely the expansion of the expression $(x+y)^n$. A binomial is a sum of two terms, usually involving unknowns, such as $(x+y)$.

**Theorem 2 (The Binomial Theorem)**

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$$

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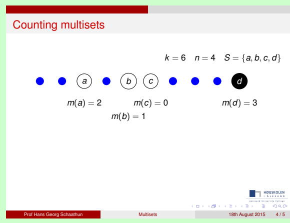
**THEORY**

A multiset is an unordered selection of elements allowing repetition. Counting the number of multisets of size $k$ from an $n$-set is an interesting application of the bijection principle.

**Related reading:** Stein *et al.* p. 80–82 or Rosen p. 139

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# 6 Compulsory assignment (Session 3)

The following problems form the compulsory assignment to be presented in Tuesday class. Please see the web pages for details. All answers must be justified.

**Problem 6.1** *You are hosting a panel discussion, with three people on the panel, named Alice, Bob, and Charlie. Write down every possible seating arrangement of these three on the three seats on the stage (from left to right, say).*

*Explain how you ensure that no combination is forgotten.*

**Problem 6.2** *How many distinct poker hands (five cards) can be drawn from a 52-card deck?*

**Problem 6.3** *The student chess club has 66 male and 37 female members. They have to elect a president and a vice-president, where the two have to have different genders. How many different combinations of president and vice-president can be formed in the club?*

**Problem 6.4** *In a tiny Chinese resturant, the chef has a repertoire consisting of three sauces (Hoisin, Szechuan, and Satay), either steamed rice, fried rice, or noodles, and either chicken or pork. A dinner should consist of one kind of meat, one source of starch (rice/noodles), and one sauce. How many distinct dishes can he put on the menu?*

**Problem 6.5** *Calculate the following*

1. $\binom{6}{2}$
2. $\binom{13}{4}$
3. $\binom{13}{9}$.
4. $\binom{905}{1}$

**Problem 6.6** *Consider two sets $M$ and $N$ of size $m$ and $n$ respectively. How many ordered pairs $(x, y)$ exist, so that $x \in M$ and $y \in N$.*