Week 2: Counting and Relations

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Period 26 August - 2 September 2015

Reading This part roughly follows Chapter 1 in Stein et al's textbook. References to Rosen's textbook have been added, but some of the concepts and terms used by Stein et al and myself will be missing from Rosen.

Programme This document details the programme for the week, including exercises and pointers to other material. It is available in two versions:

1. as a PDF document.
2. as a web site. This depends on MathML and may require firefox/iceweasel to display correctly.

The web version includes inline video. The pdf version shows a still image from the video, providing a hyperlink directly to the video on the web site.

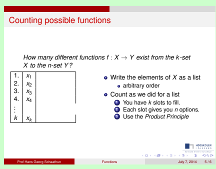
1 Session 1

1.1 Relations and functions

THEORY

Functions is a concept you have encountered both in maths and computing. Let's formalise the concept and classify different kinds of functions.

Related reading: Stein *et al.* p. 42-43 or Rosen p. 396–398



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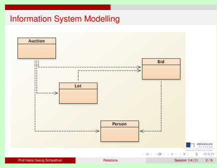
THEORY

Relations is another concept you may have encountered in computing. It also has a mathematical definition which we explore in this talk.

Related reading: Stein *et al.* p. 62 or Rosen p. 126 and 553–556



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THEORY

A function is a special case of a relation. This video explores the relationship further.

Related reading: Stein *et al.* p. 63 or Rosen p. 554

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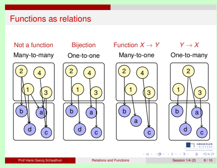
THEORY

Relations can be classified as one-to-one, many-to-one, one-to-many, or many-to-many. Let's see what signifies each of these classes

Related reading: Stein *et al.* p. 63 or Rosen p. 554



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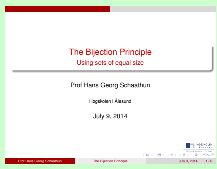
THEORY

One-to-one relations/functions are also known as bijections. The bijection principle is very useful for counting.

Related reading: Stein *et al.* p. 44–45 or Rosen p. 145 ±



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Problem 1.1 *Consider the class diagram in Figure 1. Describe each relation using common words such as* has a, contains, belongs to, *or any other phrase you deem appropriate.*

Your answer should be a list on the form

- *Class A* phrase *Class B*

Problem 1.2 *Consider weekend activities*

- *Set of activities* $A = \{\text{Horseriding}, \text{Badminton}, \text{BBQ}\}$
- *Set of days* $D = \{\text{Saturday}, \text{Sunday}\}$

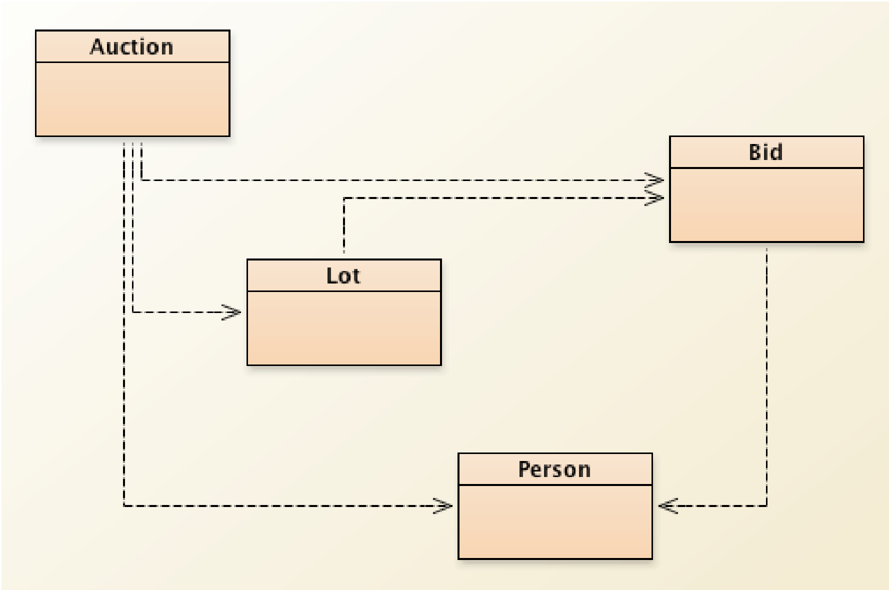


Figure 1: Class diagram.

Answer the following:

1. *List all possible functions* $A \to D$
2. *List all possible functions* $D \to A$
3. *Which of the functions are injective?*
4. *Which of the functions are surjective?*

Problem 1.3 *Consider the following relations and label them as many-to-many, one-to-many, many-to-one, or one-to-one.*

1. x *is a friend of* y
2. x *is the son of* y
3. x *is the mother of* y
4. x *is married to* y
5. x *lives at* y *(address)*

Which of the above relations can be cast as functions $x \mapsto y$*?*

Problem 1.4 *We could not possibly count the functions on real numbers, but in Discrete Mathematics we often work with finite sets. Let* T *be a* t*-element set and* S *an* s*-element set. How many functions from* S *to* T *exist?*

Problem 1.5 (Stein *et al* 1.2 Problem 17) *Explain why a function* $f : S \to S$*,* *where S is a finite (n-element) set is a bijection if and only if it is onto (surjective).*

1.2 Binomial coefficient

Problem 1.6 *What can you say in general about $\binom{n}{k}$ in relation to $\binom{n}{n-k}$?*

Problem 1.7 *Explain why*

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i} = 0$$

2 Session 2

2.1 Equivalence relations

THEORY

Equivalence relations are relations which are symmetric, reflexive, and transitive, and these have important applications to counting and other areas of mathematics.

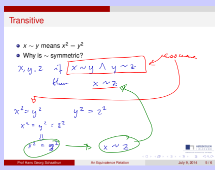
Related reading: Stein *et al.* p. 63-69, 75-76 or Rosen p. 556–558, 587

Slides

WORKED EXAMPLE

Problem 2.1 *Consider the relation $\sim$ relating x to y if $x^2 = y^2$.*

1. *Explain why $\sim$ is an equivalence relation,*
2. *Describe the equivalence classes of $\sim$.*



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Problem 2.2 *For each of the following relations, determine if they are symmetric, reflexive, or transitive.*

1. *x is an ancestor of y (x, y are people), where x is considered an ancestor of himself*
2. *x is related to y if $|x - y| \leq 3$ ($x, y \in \mathbb{Z}$)*
3. *x is related to y if $x > y + 1$ ($x, y \in \mathbb{Z}$)*
4. *(information security) x has clearance to read any file that y has access to (x, y are users of the system)*

Problem 2.3 *Which of the following relations are equivalence relations?*

1. *'Is a brother of' on the set of people*
2. *'Is a sibling of' on the set of people*
3. *'Is a sister of' on the set of women*

Justify each answer.

Problem 2.4 *Consider the set P of k-permutations of S. For every permutation t, there is a subset $T \subset S$ which contains exactly the same elements as t. We say that t corresponds to T. Two k-permutations t_1 and t_2 are* set equivalent $t_1 \sim t_2$ *if t_1 and t_2 correspond to the same subset $T \subset S$.*

Show that set equivalence is indeed an equivalence relation.

Problem 2.5 *Set equivalence as discussed in the previous problem divides P (the set of k-permutations), each class corresponding to one subset $T \subset S$ of size k. Answer the following:*

1. *What is the size of each equivalence class?*
2. *How many equivalence classes are there?*
3. *How can you use the two first answers to find the number of k-permutations?*
4. *Show how you can derive the binomial coefficient from the formula for the number of k-permutations of an n-set.*

2.2 Partial ordering

THEORY

Partial orderings form an important class of relations with many different applications.

Reading: Stein *et al* p. 69-70; Rosen p. 597-599

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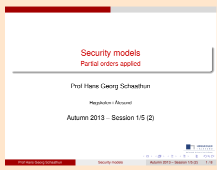
SUPPORTING THEORY

One important application of partial orderings is definition of computer security models, in particular access control models.

Reading: E.g. Dieter Gollmann: *Computer Security*



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Problem 2.6 *Consider the set of people, and the relation* is an ancestor of, *where a person is considered to be one of his own ancestors.*

- *Is this relation a partial order?*
- *Is it a total order?*

What if we consider the relation is a parent of*?*

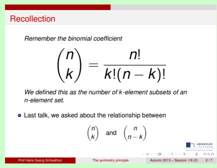
Give reasons for your answers.

Problem 2.7 *Consider the* is divisible by *relation on the set of natural numbers. I.e. we write $a \mid b$ if a/b is an integer. Is this a partial order? Is it a total order?*

3 Last lectures of the week

THEORY

Principle 1 *If a mathematical formula has a symmetry, e.g. two variable may be swapped, then a proof explaining the symmetry will usually add insight.*



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Reading: Stein *et al* p. 73-75.

4 Compulsory assignment (Tuesday Session)

Problem 4.1 *Which of the following relations are equivalence relations?*

1. *'Is a neighbour of' on the set of people living in a certain street*
2. *'Is a neighbour of' on the set of natural numbers (x and y are neighbours if $x - y = \pm 1$).*
3. *x and y related if $|x| = |y|$.*

Justify each answer.

Problem 4.2 *For each of the following relations on the set $\{1, 2, 3, 4\}$, decide whether it is (1) symmetric, (2) reflexive, and (3) transitive:*

1. $\{(2, 2), (2, 1), (1, 2), (3, 3), (3, 1), (1, 3), (4, 4), (4, 1), (1, 4)\}$.
2. $\{(2, 2), (3, 3)\}$.
3. $\{(1, 2), (2, 2), (2, 1), (1, 2), (3, 3), (4, 1), (1, 4), (4, 4)\}$.

Problem 4.3 (Rosen p. 561 ex. 3, paraphrased) *Let R be a relation on the set of web pages. Determine if R is (1) symmetric, (2) reflexive, and (3) when $(a, b) \in R$ if and only if*

1. *everyone who has visited a has also visited b.*

2. *there are no common links found on both a and b.*

3. *there is at least one common link found on both a and b.*

4. *there is some web page c with links to both a and b.*

Problem 4.4 (Stein *et al* 1.2 Problem 18) *A function $g : B \to A$ is called an* inverse *of the function $f : A \to B$, if*

a. *for every $x \in A$ we have $g(f(x)) = x$*

b. *for every $y \in B$ we have $f(g(y)) = y$*

Explain why

1. *a bijection always has an inverse function.*
2. *only bijections can have an inverse function.*
3. *a function that has an inverse has exactly one.*

Problem 4.5 (Rosen p. 402, Problem 7) *How many bit strings of 10 bits contain*

1. *exactly four ones?*
2. *at most four ones?*
3. *at least four ones?*
4. *an equal number of zeros and ones?*

Give reasons for your answers?