Exercises Week 6
Logic

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12th November 2015

Period 23–29 September 2015

Reading Stein *et al* cover this material in Chapter 3.

Programme This document details the programme for the week, including exercises and pointers to other material. It is available in two versions:

1. as a PDF document.
2. as a web site. This depends on MathML and may require firefox/iceweasel to display correctly.

The web version includes inline video. The pdf version shows a still image from the video, providing a hyperlink directly to the video on the web site.

1 Session 1

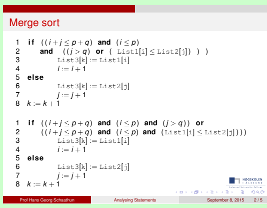
1.1 Propositional Logic

Related reading: Stein *et al.* p. 147–154 or Rosen p. 4–6 and 23–24

THEORY

We introduce propositional logic, aka. zeroth order logic. This is the simplest system of logic, with expressions using variables and the operators *and*, *or*, and *not*.

The starting point is the problem of determining whether two pieces of source code (C/Java) are equivalent.



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MP4

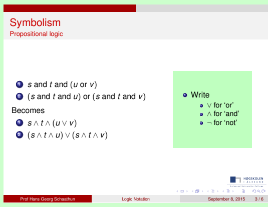
Slides

Theory

This video continues where the previous one left.

New mathematical symbols:

- $\vee$ for 'or'
- $\wedge$ for 'and'
- $\neg$ for 'not'



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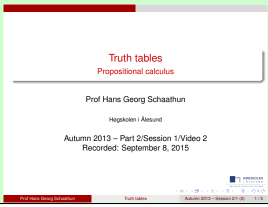
MP4

Slides

Theory

The four logic operators $\wedge$ (and), $\vee$ (or), $\oplus$ (xor), and $\neg$ (not) can be defined using truth tables.



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MP4

Slides

Exercise 1.1 *Consider $\mathbb{Z}_2$ (the integers modulo 2). Fill in the addition and multiplication tables below:*

$\cdot$	0	1
0		
1		

$+$	0	1
0		
1		

Compare these two tables to the truth tables for logic operators ($\vee$, $\wedge$, $\oplus$, $\neg$) in the video above. What is the relationship between the set $\{F, T\}$ of truth values and $\mathbb{Z}_2$? Can any of the logic operators be mapped to addition or multiplication?

Exercise 1.2 *Let F and T stand for predicates (constants) that are always false or always true respectively. Simplify the following:*

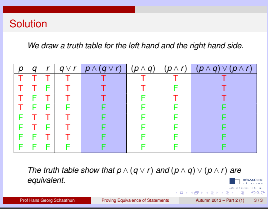
1. $s \vee s =$
2. $s \wedge s =$
3. $s \vee (\neg s) =$
4. $s \wedge (\neg s) =$
5. $T \vee s =$
6. $F \vee s =$
7. $T \wedge s =$
8. $F \wedge s =$

Worked Example

Principle 1 (The Distributive Law)

$$w \wedge (u \vee v) = (w \wedge u) \vee (w \wedge v)$$

Problem 1.1 *Prove that the distributive law is true.*



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MP4

Slides

Exercise 1.3 *There are two distributive laws. One was proved above. The other is this:*

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Prove that this law is also true, using truth tables.

1.2 Implication and equivalence

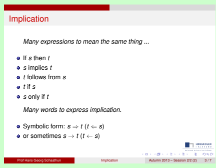
Related reading: Stein *et al.* p. 155-159 or Rosen p. 6+ and 9+

THEORY

Definition 1 *The logic operator* $\Rightarrow$ *(implication) is defined by the following truth table:*

s	t	$s \Rightarrow t$
T	T	T
T	F	F
F	T	T
F	F	T

We read $s \Rightarrow t$ *as* s implies t.



Slides

Exercise 1.4 *Show that the expression* $s \Rightarrow t$ *is equivalent to* $(\neg s) \vee (s \wedge t)$.

Definition 2 (Equivalence) *Given two predicates* s *and* t. *The notation*

$$s \Leftrightarrow t$$

means the same as

$$(s \Leftarrow t) \wedge (s \Rightarrow t),$$

and we say that s *is equivalent to* t.

Exercise 1.5 *Fill in the truth table for* $s \Leftrightarrow t$:

s	t	$s \Leftrightarrow t$
T	T	
T	F	
F	T	
F	F	

Exercise 1.6 *Give examples in plain English or Scandinavian where*

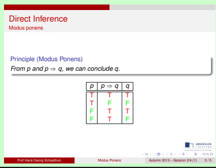
1. *«if» appears to mean «if and only if» (or where you think it would for many people).*
2. *where «if» would not mean «if and only if».*

1.3 Direct Proof

Related reading: Stein *et al.* p. 179–180 or Rosen p. 63–64

Theory

Principle 2 (Modus Ponens) *From p and $p \Rightarrow q$, we can conclude q.*



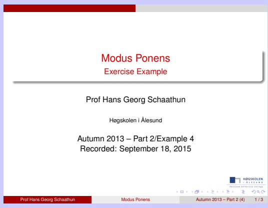
Slides

Worked Example

Exercise 1.7 *Consider the following argument*

1. *If you are clever, then you will solve this exercise.*
2. *You are clever.*
3. *Therefore you will solve this exercise.*

Rewrite the argument in symbolic form, and decide whether it is a valid argument.



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Exercise 1.8 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *If it is Wednesday, then we have fish for dinner.*
2. *It is Wednesday.*
3. *We have fish for dinner.*

Exercise 1.9 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *If it is Wednesday, then we have fish for dinner.*
2. *We have fish for dinner.*
3. *It is Wednesday.*

Exercise 1.10 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *If you fail mathematics, then you will not get your engineering degree.*
2. *You do not get your engineering degree.*
3. *Therefore you did not fail the mathematics module.*

Exercise 1.11 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *If you fail mathematics, then you will not get your engineering degree.*
2. *You fail mathematics*

3. Therefore you do not get your engineering degree.

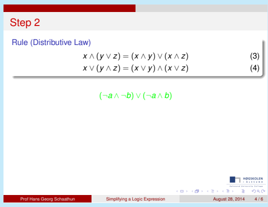
2 More examples

EXERCISE EXAMPLE

Exercise 2.1 *Simplify the following expression using deMorgan's law, the distributive law, and Exercise 1.2:*

$$\neg(a \vee b) \vee \neg(a \vee \neg b)$$

Hint: *If do not know where to start, look at the slides for the video solution without annotations and try to work out each step.*



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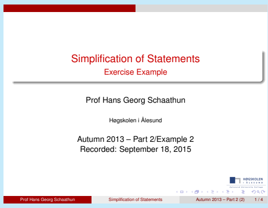
EXERCISE EXAMPLE

Exercise 2.2 *Rewrite the statement $a \Leftrightarrow b$ using only the symbols $\vee$, $\wedge$, and $\neg$ alongside a and b. What is the lowest number of symbols you can use in the rewritten expression?*



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Slides

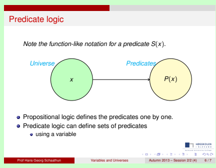
3 Session 2

3.1 Quantifiers

Related reading: Stein *et al.* p. 163–172 or Rosen p. 34–41

THEORY

Predicate logic (or first order logic) generalises propositional logic by introducing ***variables*** into the predicates.



Slides

Exercise 3.1 *Consider the predicate*

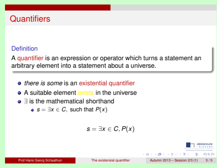
$$P(x) := |x| \leq 1$$

For what values of x is $P(x)$ true ...

1. *if the universe of x is the integers (i.e. $x \in \mathbb{Z}$)?*
2. *if the universe of x is the set of real numbers (i.e. $x \in \mathbb{R}$)?*

THEORY

Definition 3 *A* quantifier *is an expression or operator which turns a statement about an arbitrary element into a statement about a universe.*



Slides

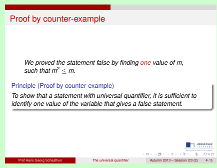
The expression *there is some* is an *existential quantifier*. Mathematically we write $\exists$ for this quantifier, and $\exists x, q(x)$ means «there exists some x such that $q(x)$ is true».

Exercise 3.2 (Video «The existential quantifier») *An equation can be thought of as a predicate. Consider the equation $x^2 + 1 = 0$.*

Express the claim that the equation has a real solution (a solution $x \in \mathbb{R}$) in symbolic form.

THEORY

Besides the existential quantifier $\exists$, we have the universal quantifier $\forall$. The expression $\forall x, q(x)$ means that «for any x, $q(x)$ is true».



Slides

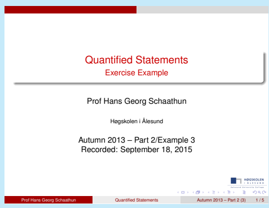
EXERCISE EXAMPLE

Exercise 3.3 *Consider the two statements*

1. *It was raining every day throughout our holiday.*
2. *There was a rain-free day during our holiday.*

Define predicate symbols and formulate the expression in symbolic form using quantifiers.

Can you see some relationship between the statements? (Implication? Equivalence? Other?)



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Exercise 3.4 *A natural number p is prime if it has no positive integer factor other than one and itself. Find two ways to write this criterion in symbolic form. You need to introduce a variable x to denote potential factors of p, and use either the existential or universal quantifier on x for two different ways.*

Each answer could begin with the words «$p \in \mathbb{N}$ is a prime number if and only if …».

Exercise 3.5 *Which of the following statements are true and which are false?*

1. $\forall z \in \mathbb{N},\ z^2 + 5z + 10 \geq 16$
2. $\exists z \in \mathbb{N},\ z^2 < 1$
3. $\exists z \in \mathbb{Z},\ z^2 < 1$

The following definitions are used

$$\mathbb{Z} = \{\ldots, -1, 0, +1, \ldots\} \quad \textit{the integers} \tag{1}$$
$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \textit{the natural numbers} \tag{2}$$
$$\mathbb{R} \textit{ is the set of real numbers} \tag{3}$$
$$\mathbb{C} \textit{ is the set of complex numbers} \tag{4}$$

Exercise 3.6 *What is the difference between the two following statements?*

$$\forall x \in \mathcal{X}, q(x), \tag{5}$$
$$x \in \mathcal{X} \Longrightarrow q(x) \tag{6}$$

Discuss. Is there a difference at all?

4 Last lectures

Related reading: Stein *et al.* p. 179–181 or Rosen p. 63–64, 71–72, and 85±

THEORY

One common pitfall in logic arguments is to confuse the validity of the argument with whether the conclusion is true or false. These are totally independent matters; let's discuss.

Slides

THEORY

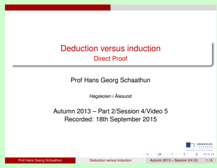
Sometimes we need to prove a statement of the form a implies b.

Principle 3 (Conditional proof) *If, by assuming just that p is true, we can prove q, then we can conclude that $p \Rightarrow q$.*

Slides

THEORY

Another common pitfall in logic arguments is false generalisation, also known as *induction*. Let's discuss this concept of induction.



Slides

5 Compulsory Assignment (Tuesday 22 September 2015)

Exercise 5.1 *Prove DeMorgan's law which states that*

$$\neg(s \wedge t) = \neg s \vee \neg t$$

You can use the same technique as in the previous example.

Exercise 5.2 *Give truth tables for the following:*

1. $(s \vee t) \wedge (\neg s \vee t) \wedge (s \vee \neg t)$

Exercise 5.3 *Consider the following statement:*

> *Alice and Bob are not both ill.*

1. *Define predicate symbols and rewrite the statement in symbolic form.*
2. *Use deMorgan's law to rephrase the statement.*
3. *Rephrase the statement in English (or Norwegian), using the word 'well' rather than 'ill' without changing the meaning..*

Exercise 5.4 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *An equilateral triangle (defined as one with three edges of equal length), has three equal angles.*
2. *Triangle T has three angles of 60° each.*

3. *Therefore T is equilateral.*

Exercise 5.5 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *A triangle is equilateral (defined as above) if and only if it has three equal angles.*
2. *Triangle T has three angles of 60° each.*
3. *Therefore T is equilateral.*

Exercise 5.6 *Is the following a valid argument?*

- *The whale is a fish.*
- *All fish can swim.*
- *Therefore the whale can swim.*

Exercise 5.7 *Is the following a valid argument?*

- *The whale is not a fish.*
- *All fish can swim.*
- *Therefore the whale cannot swim.*

Exercise 5.8 *Which of the following statements are true and which are false?*

1. $\forall z \in \mathbb{Z},\ z^2 \geq z$
2. $\forall z \in \mathbb{R},\ z^2 \geq z$
3. $\exists z \in \mathbb{R},\ z^2 = -1$
4. $\exists z \in \mathbb{C},\ z^2 = -1$