Solutions Week 6
Logic

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This document includes all the exercises from the main programme, some of them with solutions. It is available both as a PDF document and as a web site.

The main programme can also be found either as a PDF document and as a web site.

Note that the web versions use MathML and may require Firefox to display correctly.

1 Session 1

1.1 Propositional Logic

Related reading: Stein *et al.* p. 147–154 or Rosen p. 4–6 and 23–24

Exercise 1.1 *Consider $\mathbb{Z}_2$ (the integers modulo 2). Fill in the addition and multiplication tables below:*

$\cdot$	0	1
0		
1		

$+$	0	1
0		
1		

Compare these two tables to the truth tables for logic operators ($\vee$, $\wedge$, $\oplus$, $\neg$) in the video above. What is the relationship between the set $\{F, T\}$ of truth values and $\mathbb{Z}_2$? Can any of the logic operators be mapped to addition or multiplication?

Exercise 1.2 *Let F and T stand for predicates (constants) that are always false or always true respectively. Simplify the following:*

1. $s \vee s =$

2. $s \wedge s =$

3. $s \vee (\neg s) =$

4. $s \wedge (\neg s) =$

5. $T \vee s =$

6. $F \vee s =$

7. $T \wedge s =$

8. $F \wedge s =$

SOLUTION:

1. $s \vee s = s$
2. $s \wedge s = s$
3. $s \vee (\neg s) = T$
4. $s \wedge (\neg s) = F$

1. $T \vee s = T$
2. $F \vee s = s$
3. $T \wedge s = s$
4. $F \wedge s = F$

Exercise 1.3 *There are two distributive laws. One was proved above. The other is this:*

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$$

Prove that this law is also true, using truth tables.

1.2 Implication and equivalence

Related reading: Stein *et al.* p. 155-159 or Rosen p. 6+ and 9+

Exercise 1.4 *Show that the expression $s \Rightarrow t$ is equivalent to $(\neg s) \vee (s \wedge t)$.*

Definition 1 (Equivalence) *Given two predicates s and t. The notation*

$$s \Leftrightarrow t$$

means the same as

$$(s \Leftarrow t) \wedge (s \Rightarrow t),$$

and we say that s is equivalent to t.

Exercise 1.5 *Fill in the truth table for $s \Leftrightarrow t$:*

s	t	$s \Leftrightarrow t$
T	T	
T	F	
F	T	
F	F	

Exercise 1.6 *Give examples in plain English or Scandinavian where*

1. *«if» appears to mean «if and only if» (or where you think it would for many people).*
2. *where «if» would not mean «if and only if».*

SOLUTION: For instance:

1. If the sun is shining, then I go swimming.
2. If you fail mathematics, then you will not get your degree.

1.3 Direct Proof

Related reading: Stein *et al.* p. 179–180 or Rosen p. 63–64

Exercise 1.7 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *If it is Wednesday, then we have fish for dinner.*
2. *It is Wednesday.*
3. *We have fish for dinner.*

Exercise 1.8 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *If it is Wednesday, then we have fish for dinner.*
2. *We have fish for dinner.*
3. *It is Wednesday.*

Exercise 1.9 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *If you fail mathematics, then you will not get your engineering degree.*
2. *You do not get your engineering degree.*
3. *Therefore you did not fail the mathematics module.*

Exercise 1.10 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *If you fail mathematics, then you will not get your engineering degree.*
2. *You fail mathematics*
3. *Therefore you do not get your engineering degree.*

2 More examples

3 Session 2

3.1 Quantifiers

Related reading: Stein *et al.* p. 163–172 or Rosen p. 34–41

Exercise 3.1 *Consider the predicate*

$$P(x) := |x| \leq 1$$

For what values of x is $P(x)$ true ...

1. *if the universe of x is the integers (i.e. $x \in \mathbb{Z}$)?*
2. *if the universe of x is the set of real numbers (i.e. $x \in \mathbb{R}$)?*

SOLUTION:

1. It is true for $x \in \{-1, 0, +1\}$ (discrete set).
2. It is true for $x \in [-1, +1]$ (closed interval).

Exercise 3.2 (Video «The existential quantifier») *An equation can be thought of as a predicate. Consider the equation $x^2 + 1 = 0$.*

Express the claim that the equation has a real solution (a solution $x \in \mathbb{R}$) in symbolic form.

SOLUTION:

$$\exists x \in \mathbb{R}, x^2 + 1 = 0$$

Exercise 3.3 *A natural number p is prime if it has no positive integer factor other than one and itself. Find two ways to write this criterion in symbolic form. You need to introduce a variable x to denote potential factors of p, and use either the existential or universal quantifier on x for two different ways.*

Each answer could begin with the words «$p \in \mathbb{N}$ is a prime number if and only if ...».

Solution:

1. p is prime number if
$$\forall x \in \{2, 3, \ldots, p-1\}, x \not| p.$$

2. p is prime number if
$$\neg \exists x \in \{2, 3, \ldots, p-1\}, x \mid p.$$

Exercise 3.4 *Which of the following statements are true and which are false?*

1. $\forall z \in \mathbb{N},\ z^2 + 5z + 10 \geq 16$
2. $\exists z \in \mathbb{N},\ z^2 < 1$
3. $\exists z \in \mathbb{Z},\ z^2 < 1$

The following definitions are used

$$\mathbb{Z} = \{\ldots, -1, 0, +1, \ldots\} \quad \textit{the integers} \tag{1}$$
$$\mathbb{N} = \{1, 2, 3, \ldots\} \quad \textit{the natural numbers} \tag{2}$$
$$\mathbb{R} \textit{ is the set of real numbers} \tag{3}$$
$$\mathbb{C} \textit{ is the set of complex numbers} \tag{4}$$

Exercise 3.5 *What is the difference between the two following statements?*

$$\forall x \in \mathcal{X}, q(x), \tag{5}$$
$$x \in \mathcal{X} \Longrightarrow q(x) \tag{6}$$

Discuss. Is there a difference at all?

4 Last lectures

Related reading: Stein *et al.* p. 179–181 or Rosen p. 63–64, 71–72, and 85±

5 Compulsory Assignment (Tuesday 22 September 2015)

Exercise 5.1 *Prove DeMorgan's law which states that*

$$\neg(s \wedge t) = \neg s \vee \neg t$$

You can use the same technique as in the previous example.

Exercise 5.2 *Give truth tables for the following:*

1. $(s \vee t) \wedge (\neg s \vee t) \wedge (s \vee \neg t)$

Exercise 5.3 *Consider the following statement:*

Alice and Bob are not both ill.

1. *Define predicate symbols and rewrite the statement in symbolic form.*
2. *Use deMorgan's law to rephrase the statement.*
3. *Rephrase the statement in English (or Norwegian), using the word 'well' rather than 'ill' without changing the meaning..*

Exercise 5.4 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *An equilateral triangle (defined as one with three edges of equal length), has three equal angles.*
2. *Triangle T has three angles of 60° each.*
3. *Therefore T is equilateral.*

Exercise 5.5 *Rewrite the following argument in symbolic form, and decide whether or not it is a valid argument.*

1. *A triangle is equilateral (defined as above) if and only if it has three equal angles.*
2. *Triangle T has three angles of 60° each.*
3. *Therefore T is equilateral.*

Exercise 5.6 *Is the following a valid argument?*

- *The whale is a fish.*
- *All fish can swim.*
- *Therefore the whale can swim.*

Exercise 5.7 *Is the following a valid argument?*

- *The whale is not a fish.*
- *All fish can swim.*
- *Therefore the whale cannot swim.*

Exercise 5.8 *Which of the following statements are true and which are false?*

1. $\forall z \in \mathbb{Z},\ z^2 \geq z$
2. $\forall z \in \mathbb{R},\ z^2 \geq z$

3. $\exists z \in \mathbb{R},\ z^2 = -1$
4. $\exists z \in \mathbb{C},\ z^2 = -1$