Upper bounds on Weight Hierarchies of Extremal Non-Chain Codes

Hans Georg Schaathun

Department of Informatics, Universitas Bergensis, Høyteknologisenteret, N-5020 Bergen, Norway

Abstract

The weight hierarchy of a linear $[n, k; q]$ code $\mathcal{C}$ over $\mathsf{GF}(q)$ is the sequence $(d_1, d_2, \dots, d_k)$ where d_r is the smallest support weight of an r-dimensional subcode of $\mathcal{C}$. Linear codes may be classified according to a set of chain and non-chain conditions, the extreme cases being codes satisfying the chain condition (due to Wei and Yang) and extremal, non-chain codes (due to Chen and Kløve). This paper gives upper bounds on the weight hierarchies of the latter class of codes.

Key words: Weight hierarchy, chain condition, linear codes, projective multiset

1 Introduction

The concept of generalised Hamming weights was introduced as early as 1977 by Helleseth et al. [8] in their study of weight distributions of irreducible cyclic codes. The term 'generalised Hamming weight' was introduced by Wei in 1991 [14]. He used the parameters to analyse an application of codes on the Wire-Tap Channel of type II, which had been introduced in 1984 by Ozarow and Wyner [11]. During the nineties, several researchers have studied the generalised Hamming weights of linear codes.

The chain condition was introduced by Wei and Yang [15]. Chen and Kløve [2] introduced the opposite extreme, extremal non-chain codes. Known codes with high generalised Hamming weights tend to satisfy the chain condition. Cohen et al. [6] argue that some non-chain codes may have other advantages. Our interest is purely mathematical however.

Chen and Kløve found tight upper bounds for non-binary, four-dimensional, extremal non-chain codes [2]. Later they have also found all possible weight hierarchies of four-dimensional binary codes [5]. In this paper we generalise their upper bounds to arbitrary dimension, and these bounds are the best possible in dimension 5 and lower.

1.1 Notation and definitions

Throughout this paper $\mathcal{C}$ will denote an $[n, k+1; q]$ code, i.e. a linear code of length n and dimension $k+1$ over the Galois field $\mathsf{GF}(q)$ with q elements. Codes of dimension $k+1$ will be studied in a projective space $\mathsf{PG}(k, q)$ of dimension k and order q.

Given a code $\mathcal{C}$ we define the support $\chi(\mathcal{C})$ to be the set of positions where not all codewords of $\mathcal{C}$ are zero, i.e.

$$\chi(\mathcal{C}) := \{i \mid \exists(x_1, x_2, \ldots, x_n) \in \mathcal{C}, \ \text{ s.t. } x_i \neq 0\}.$$

The *support weight* of $\mathcal{C}$ is the size of $\chi(\mathcal{C})$, and we denote it $w_S(\mathcal{C})$, i.e.

$$w_S(\mathcal{C}) := \#\chi(\mathcal{C}).$$

For $0 \leq r \leq k+1$, *the rth generalised Hamming weight* d_r of $\mathcal{C}$ is the least support weight of an r-dimensional subcode of $\mathcal{C}$. The sequence $(d_1, d_2, \ldots, d_{k+1})$ is called the weight hierarchy of $\mathcal{C}$. The minimum weight of the code is $d = d_1$.

We note that by adding a zero-position to $\mathcal{C}$, we get an $[n+1, k+1; q]$ code with the same weight hierarchy as $\mathcal{C}$. Without loss of generality, we can restrict our study to codes without zero-positions. In other words, we assume that $d_{k+1} = n$.

Two linear codes are *equivalent* if one can be obtained from the other by permuting coordinate positions or by multiplying some coordinate by a non-zero scalar. We note that equivalent codes have the same weight hierarchy.

1.2 Codes in projective geometry

We let G denote a generator matrix of $\mathcal{C}$. The *value* (or multiplicity) $\nu(\mathbf{x})$ of $\mathbf{x} \in \mathsf{GF}(q)^{k+1}$ is the number of occurrences of $\mathbf{x}$ as a column in G. Replacing some column $\mathbf{x}$ with $a\mathbf{x}$ for some non-zero scalar a leads to an equivalent code. Thus we can consider the columns of G to be projective points, and an equivalence class of codes is uniquely determined by giving the map

$$\nu : \mathsf{PG}(k, q) \longrightarrow \mathbb{N}_0 := \{0, 1, \ldots\}.$$

This concept has been studied by several authors using different terminology. Dodunekov and Simonis [7] give an historic overview, and they prefer to call ν a projective multiset. In this paper we prefer to call it a *value assignment*, as did Chen and Kløve [2]. Tsfasman and Vladuţ [13] studied an equivalent concept called a projective system.

An arbitrary map $\nu : \mathsf{PG}(k,q) \longrightarrow \mathbb{N}_0$ is called a value assignment even if it is not defined from a code. We call it *non-degenerate* if there are $k+1$ projectively independent points $p_0, p_1, \ldots, p_k$ such that $\nu(p_i) \geq 1$ for all i. By taking $\nu(\mathbf{x})$ not necessarily distinct representatives for each projective point and taking an ordering on all these representatives, we get a matrix G. This matrix G is a generator matrix of a code if and only if its rank is $k+1$, that is if ν is non-degenerate.

We define the value of a set of points as follows

$$\nu(U) := \sum_{\mathbf{x} \in U} \nu(\mathbf{x}), \quad \forall\, U \subseteq \mathsf{PG}(k,q).$$

Let $\mathsf{PG}^{(m)}(k,q)$ be the set of m*-spaces* or m-dimensional subspaces of $\mathsf{PG}(k,q)$. Note that $\mathsf{PG}^{(0)}(k,q)$ is the collection of subsets of cardinality 1; both $P \in \mathsf{PG}(k,q)$ and $\{P\} \in \mathsf{PG}^{(0)}(k,q)$ will be called a point. The 1-, 2- and $(k-1)$-spaces are called lines, planes, and hyperplanes, respectively. The only (-1)-space is the empty set.

The join of Π_r and Π_s, denoted $\Pi_r\Pi_s$, is the intersection of all subspaces containing the union $\Pi_r \cup \Pi_s$. If $p_0, p_1, \ldots, p_m \in \mathsf{PG}(k,q)$ are projectively independent points, we write $\langle p_0, p_1, \ldots, p_m \rangle$ for their join. We define the following shorthand notation,

$$\theta(n) := \frac{q^{n+1}-1}{q-1} = \sum_{i=0}^{n} q^i,$$

and recall that $\theta(k)$ is the cardinality of $\mathsf{PG}(k,q)$.

1.3 Subcodes and the value assignments

From now on we let $\nu : \mathsf{PG}(k,q) \rightarrow \mathbb{N}_0$ be the value assignment corresponding to $\mathcal{C}$. There is a one-to-one correspondence between subcodes of $\mathcal{C}$ of dimension r and subspaces of $\mathsf{PG}(k,q)$ of dimension $k-r$. We write $\mathcal{D}^*$ for the projective subspace corresponding to a subcode $\mathcal{D} \subseteq \mathcal{C}$, and Π^* for the subcode corresponding to $\Pi \subseteq \mathsf{PG}(k,q)$. If $\mathcal{D}_1 \subseteq \mathcal{D}_2$, then $\mathcal{D}_1^* \supseteq \mathcal{D}_2^*$. It is known [9,13] that $d_{k+1} - w_S(\mathcal{D}) = \nu(\mathcal{D}^*)$.

We define the weight hierarchy $(d_1, \ldots, d_{k+1})$ of a value assignment ν by letting

$n - d_r$ be the greatest value of a subspace of codimension r in $\mathsf{PG}(k,q)$. Obviously the correspondence between value assignments and codes preserves the weight hierarchy. Note that a value assignment is non-degenerate if and only if $d_1 > 0$. All value assignments encountered in this paper are non-degenerate.

The difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ of a code or of a value assignment is defined by

$$\delta_j := d_{k+1-j} - d_{k-j}, \quad j = 0, 1, \ldots, k.$$

We note that the difference sequence is easily computed from the weight hierarchy and vice versa. We say that the difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ has dimension $k+1$. The elements of the difference sequence of a code or non-degenerate value assignment are positive, due to the strict monotonicity of the generalised Hamming weights.

The existence of a linear code with weight hierarchy $(d_1, d_2, \ldots, d_{k+1})$ is equivalent to the existence of a non-degenerate value assignment ν such that,

$$\max\{\nu(\Pi_m) \mid \Pi_m \in \mathsf{PG}^{(m)}(k,q)\} = \sum_{i=0}^{m} \delta_i, \quad -1 \le m \le k.$$

The set of m-spaces of maximum value is denoted by M_m,

$$M_m(\nu) := \left\{ \Pi_m \;\middle|\; \Pi_m \in \mathsf{PG}^{(m)}(k,q) \;\wedge\; \nu(\Pi_m) = \sum_{i=0}^{m} \delta_i \right\}, \quad -1 \le m \le k.$$

When no ambiguity is expected, we write $M_m = M_m(\nu)$.

Given an m-space $\Pi_m \in \mathsf{PG}^{(m)}(k,q)$, we can restrict the value assignment ν to this subspace and study

$$\nu' = \nu|_{\Pi_m} : \; \Pi_m \to \mathbb{N}_0.$$

If $\Pi_m \in M_m(\nu)$, the monotonicity of the weight hierarchy ensures that any proper subspace of Π_m has lower value. In this case ν' is non-degenerate, and thus defines a code $\mathcal{D}$, which is actually the code obtained by puncturing $\mathcal{C}$ on each coordinate in $\chi(\Pi_m^*)$. We write $M_i(\Pi_m)$ $M_i(\nu|_{\Pi_m})$ for $-1 \le i \le m$.

1.4 The Chain Condition

The chain condition was introduced by Wei and Yang [15], and it says that

$$\forall i \text{ s.t. } 0 \le i \le k-1 \quad \exists \Pi_i \in M_i \quad \text{s.t. } \Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{k-1}.$$

We will refer to codes satisfying this condition as *chained codes*.

We define a number of subconditions, which are implications of the chain condition. For all i and j such that $0 \leq i < j \leq k-1$, we have the condition,

$$(\mathrm{C}i.j): \quad \exists \Pi_i \in M_i \, \exists \Pi_j \in M_j \text{ s.t. } \Pi_i \subset \Pi_j.$$

The negations of these conditions, $(\mathrm{N}i.j) := \neg(\mathrm{C}i.j)$, will be called *non-chain conditions*.

Analogous to the definition by Chen and Kløve [2], we define extremal non-chain codes of arbitrary dimension to be codes that satisfy all of the non-chain conditions $(\mathrm{N}i.j)$. The difference sequence of an extremal non-chain code will be called an ENDS (*extremal non-chain difference sequence*).

2 Upper bounds

2.1 The general upper bound

Theorem 1 *If $(\delta_0, \delta_1, \ldots \delta_k)$ is an ENDS and $1 \leq m \leq k-1$, then*

$$\delta_m \leq q^m \delta_0 - \sum_{i=0}^{m} q^i.$$

If this bound holds with equality for $m = \bar{m} > 1$, then it also holds with equality for $m = \bar{m} - 1$.

The proof of this theorem is quite tedious, and we have to start with some auxiliary results.

Definition 2 *We say that an ENDS is* m-optimal, $1 \leq m \leq k-1$, *if it satisfies the bound on δ_m from Theorem 1 with equality. An extremal non-chain code C is* m-optimal *if its difference sequence is an m-optimal ENDS.*

Lemma 3 *Given an arbitrary code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$, we have $\delta_k \leq q\delta_{k-1}$.*

PROOF. Take some $\Pi_{k-2} \in M_{k-2}$. There are $q+1$ $(k-1)$-spaces through Π_{k-2}, and for every such subspace Π_{k-1} we have

$$\nu(\Pi_{k-1} \backslash \Pi_{k-2}) \leq \delta_{k-1}.$$

The geometry is partitioned into $q+1$ disjoint subsets of the form $\Pi_{k-1} \backslash \Pi_{k-2}$, beside Π_{k-2}. Hence

$$\sum_{i=0}^{k} \delta_i \leq (q+1)\delta_{k-1} + \sum_{i=0}^{k-2} \delta_i.$$

The lemma follows immediately. $\square$

Lemma 4 *Let $(\delta_0, \delta_1, \dots \delta_k)$ be the difference sequence of some non-degenerate value assignment ν, and $(\delta'_0, \dots, \delta'_{k-1})$ the difference sequence of $\nu|_{\Pi_{k-1}}$ for some $\Pi_{k-1} \in M_{k-1}$. Then $\delta_{k-1} \leq \delta'_{k-1}$.*

PROOF. We have $\Pi_{k-1} \in M_{k-1}(\Pi_{k-1}) \subseteq M_{k-1}(\nu)$. Let $\Pi_{k-2} \in M_{k-2}(\nu)$ and $\Pi'_{k-2} \in M_{k-2}(\Pi_{k-1})$. Clearly $\nu(\Pi'_{k-2}) \leq \nu(\Pi_{k-2})$. Hence

$$\delta_{k-1} = \nu(\Pi_{k-1}) - \nu(\Pi_{k-2}) \leq \nu(\Pi_{k-1}) - \nu(\Pi'_{k-2}) = \delta'_{k-1},$$

as required. $\square$

Lemma 5 *Let ν be the value assignment of an extremal non-chain code with difference sequence $(\delta_0, \delta_1, \dots \delta_k)$. If $\Pi_m \in M_m$ where $0 \leq m \leq k$ and $\nu|_{\Pi_m}$ has difference sequence $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_m)$, then $\delta_m \leq \varepsilon_m - 1$.*

PROOF. This goes almost like the proof of Lemma 4, except that since the code is extremal non-chain, we get a stronger bound. We have $\Pi_m \in M_m(\Pi_m) \subseteq M_m(\nu)$. Let $\Pi_{m-1} \in M_{m-1}(\nu)$ and $\Pi'_{m-1} \in M_{m-1}(\Pi_m)$. Since the code is extremal non-chain, we have $\nu(\Pi'_m) < \nu(\Pi_m)$. Hence

$$\delta_m = \nu(\Pi_m) - \nu(\Pi_{m-1}) \leq \nu(\Pi_m) - (\nu(\Pi'_{m-1}) + 1) = \varepsilon_m - 1,$$

as required. $\square$

Lemma 6 *If $k \geq 2$ and $(\delta_0, \delta_1, \dots \delta_k)$ satisfies* (N0.1)*, then $\delta_1 \leq q\delta_0 - (q+1)$ and $\delta_0 \geq 2$.*

PROOF. A line consists of $q+1$ points, and by (N0.1), $\delta_1 + \delta_0 \leq (q+1)(\delta_0 - 1)$. Hence $\delta_1 \leq q\delta_0 - (q+1)$. Also if $\delta_0 \leq 1$, then $\delta_1 \leq -1$, which is absurd. $\square$

Proof of Theorem 1. The proof goes by induction on m, so we assume that the theorem holds for every $m < \bar{m}$. Lemma 6 proves it for $m = 1$. Now we consider a code $\mathcal{C}$ such that

$$\delta_{\bar{m}} \geq q^{\bar{m}}\delta_0 - \theta(\bar{m}) \tag{1}$$

$$\delta_m \leq q^m\delta_0 - \theta(m), \quad \forall m \leq \bar{m} - 1. \tag{2}$$

Our aim is to prove that then we must have equality both in (1) and in (2).

Take an arbitrary $\Theta_{\bar{m}} \in M_{\bar{m}}(\mathcal{C})$, and let

$$\Theta_0 \subset \Theta_1 \subset \ldots \subset \Theta_{\bar{m}-1} \subset \Theta_{\bar{m}}$$

be a chain such that $\Theta_i \in M_i(\Theta_{i+1})$ for $0 \leq i \leq \bar{m}-1$. Let $(\varepsilon_0^{(i)}, \ldots, \varepsilon_i^{(i)})$ be the difference sequence of $\nu|_{\Theta_i}$.

By Lemma 5 and (1), we get

$$\varepsilon_{\bar{m}}^{(\bar{m})} \geq \delta_{\bar{m}} + 1 \geq q^{\bar{m}}\delta_0 - \theta(\bar{m}) + 1. \tag{3}$$

Lemma 4 and 3 give

$$\varepsilon_{\bar{m}-1}^{(\bar{m}-1)} \geq \varepsilon_{\bar{m}-1}^{(\bar{m})} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q} \right\rceil. \tag{4}$$

Repeating this argument $\bar{m}$ times and substituting from (3), we obtain

$$\varepsilon_0^{(0)} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q^{\bar{m}}} \right\rceil \geq \left\lceil \frac{q^{\bar{m}}\delta_0 - \theta(\bar{m}) + 1}{q^{\bar{m}}} \right\rceil = \delta_0 - 1.$$

Clearly $\varepsilon_0^{(0)}$ is the value of Θ_0, which is a point in $\Theta_{\bar{m}} \in M_{\bar{m}}(\mathcal{C})$. Since $\mathcal{C}$ is extremal non-chain, we have $\varepsilon_0^{(0)} \leq \delta_0 - 1$. We conclude that

$$\varepsilon_0^{(l)} = \nu(\Theta_0) = \delta_0 - 1, \quad \forall l,\ 0 \leq l \leq \bar{m}. \tag{5}$$

We assume for induction on j that for all $j < i$ where $0 < i < \bar{m}$, we have

$$\varepsilon_j^{(l)} = \varepsilon_j^{(j)} = q^j\delta_0 - \theta(j), \quad \forall l, \text{ s.t. } j \leq l \leq \bar{m}. \tag{6}$$

First we prove that it also holds for $l = j = i$. Repeating the argument of (4) $\bar{m} - i$ times, we get

$$\varepsilon_i^{(i)} \geq \left\lceil \frac{\varepsilon_{\bar{m}}^{(\bar{m})}}{q^{\bar{m}-i}} \right\rceil \geq \left\lceil \frac{q^{\bar{m}}\delta_0 - \theta(\bar{m}) + 1}{q^{\bar{m}-i}} \right\rceil = q^i\delta_0 - \theta(i). \tag{7}$$

Now $\varepsilon_i^{(i)} = \nu(\Theta_i) - \nu(\Theta_{i-1})$. Since $\mathcal{C}$ is extremal non-chain, we get by (2) that

$$\nu(\Theta_i) \leq \sum_{j=0}^{i} \delta_j - 1 \leq \sum_{j=0}^{i} [q^j\delta_0 - \theta(j)],$$

and according to the induction hypothesis (6), we have

$$\nu(\Theta_{i-1}) = \sum_{j=0}^{i-1} \varepsilon_j^{(i-1)} = \sum_{j=0}^{i-1} \varepsilon_j^{(j)} = \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)]. \tag{8}$$

Combining these expressions, we get an upper bound on $\varepsilon_i^{(i)}$:

$$\begin{aligned} \varepsilon_i^{(i)} &= \nu(\Theta_i) - \nu(\Theta_{i-1}) \\ &\leq \sum_{j=0}^{i} [q^j \delta_0 - \theta(j)] - \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)] = q^i \delta_0 - \theta(i). \end{aligned} \tag{9}$$

Combining the upper and lower bounds (7) and (9), we conclude by induction that

$$\varepsilon_i^{(i)} = q^i \delta_0 - \theta(i), \quad i = 0, \dots, \bar{m} - 1. \tag{10}$$

From (8) and (2) we can see that

$$\sum_{j=0}^{i-1} \delta_j - 1 \geq \nu(\Theta_{i-1}) = \sum_{j=0}^{i-1} [q^j \delta_0 - \theta(j)] \geq \sum_{j=0}^{i-1} \delta_j - 1,$$

Hence $\delta_{i-1} = q^{i-1} \delta_0 - \theta(i-1)$. Also

$$\varepsilon_i^{(i)} + \nu(\Theta_{i-1}) = q^i \delta_0 - \theta(i) + \nu(\Theta_{i-1}) = \nu(\Theta_i) \leq \sum_{j=0}^{i} \delta_j - 1.$$

Hence $q^i \delta_0 - \theta(i) \leq \delta_i$, and combining with (2), we get $\delta_i = q^i \delta_0 - \theta(i)$.

It follows from this argument that $\Theta_i \in M_i(\Theta_l)$ and $\Theta_{i-1} \in M_{i-1}(\Theta_l)$, for all l such that $i \leq l \leq \bar{m}$, and hence $\varepsilon_i^{(l)} = \varepsilon_i^{(i)}$. It follows by induction that $\delta_i = q^i \delta_0 - \theta(i)$ for $i = 1, 2, \dots, \bar{m} - 1$.

We have

$$\varepsilon_{\bar{m}}^{(\bar{m})} = \nu(\Theta_{\bar{m}}) - \nu(\Theta_{\bar{m}-1}) = \sum_{i=0}^{\bar{m}} \delta_i - \left(\sum_{i=0}^{\bar{m}-1} \delta_i - 1 \right) = \delta_{\bar{m}} + 1,$$

and by Lemmas 3 and 4 and (10), we get

$$\delta_{\bar{m}} + 1 = \varepsilon_{\bar{m}}^{(\bar{m})} \leq q \varepsilon_{\bar{m}-1}^{(\bar{m})} \leq q \varepsilon_{\bar{m}-1}^{(\bar{m}-1)} = q^{\bar{m}} \delta_0 - \theta(\bar{m}) + 1.$$

Combining with the lower bound from (1) we get

$$\delta_{\bar{m}} = \varepsilon_{\bar{m}}^{(\bar{m})} - 1 = q^{\bar{m}} \delta_0 - \theta(\bar{m}),$$

and the theorem follows by induction. □

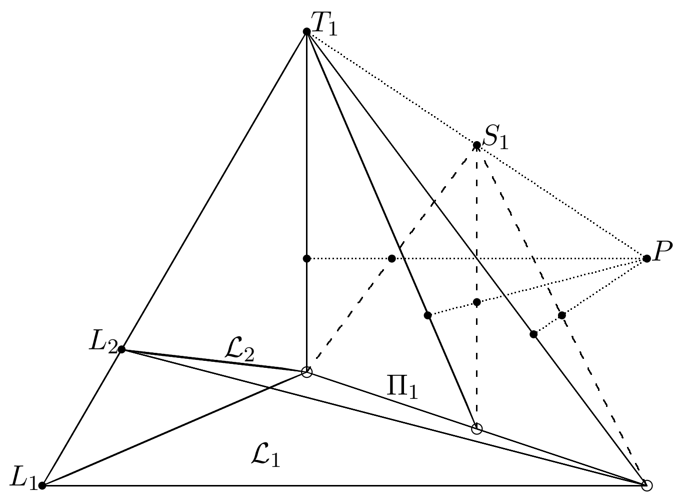

Figure 1. Representation of PG(4, 2) for the proof of Theorem 9. Black lines are in Π_3, dashed lines in Π_2, and dotted lines are in neither. White points are in Π_1. The point L_1 and Π_1 span $\mathcal{L}_1$, and L_2 and Π_1 span $\mathcal{L}_2$.

Corollary 7 *Let $\mathcal{C}$ be an m-optimal code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ for some m such that $1 \leq m \leq k-1$. For every $\Pi_m \in M_m$, $\nu|_{\Pi_m}$ corresponds to a chained code with difference sequence $(\delta_0 - 1, \delta_1, \delta_2, \ldots, \delta_{m-1}, \delta_m + 1)$.*

PROOF. In the proof of Theorem 1 we proved that $\Theta_i \in M_i(\Theta_m) = M_i(\Pi_m)$, and we found the difference sequence as given in the corollary. □

Remark 8 *We know from Lemma 6 that $\delta_0 \geq 2$, so the difference sequence has only positive elements as expected. Writing*

$$(\varepsilon_0 = \delta_0 - 1, \varepsilon_1 = \delta_1, \ldots, \varepsilon_{m-1} = \delta_{m-1}, \varepsilon_m = \delta_m + 1)$$

for the difference sequence of $\nu|_{\Pi_m}$, we have $\varepsilon_i = q\varepsilon_{i-1} - 1$ for $i = 1, \ldots, m-1$ and $\varepsilon_m = q\varepsilon_{m-1}$.

2.2 Binary case

For binary codes we have a special bound, which also implies that binary codes cannot be $(k-1)$-optimal if $k \geq 3$.

Theorem 9 *If $(\delta_0, \delta_1, \ldots \delta_k)$, $k \geq 3$, is a binary ENDS, then*

$$\delta_{k-1} \leq 2^{k-2}\delta_1 - 2 - 2^{k-2}.$$

PROOF. Take $\Pi_{k-1} \in M_{k-1}$ and $\Pi_{k-2} \in M_{k-2}$, and let

$$\Pi_{k-3} Pi_{k-1} \cap \Pi_{k-2}.$$

Because the code is extremal non-chain, Π_{k-3} is a $(k-3)$-space. Also let $\{P\} \in M_0$.

Define

$$\begin{aligned} \mathcal{S} &:= \Pi_{k-2} \backslash \Pi_{k-3} = \{S_i \mid i = 1, 2, \ldots, 2^{k-2}\}, \\ \ell_i &:= \langle P, S_i \rangle = \{P, S_i, T_i\}, \quad i = 1, 2, \ldots, 2^{k-2}. \end{aligned}$$

Every line through P meets Π_{k-1}, so the points T_i are in Π_{k-1}. Define the set

$$\mathcal{T} := \{T_i \mid i = 1, 2, \ldots, 2^{k-2}\}.$$

Because the code is an ENDS, $\nu(\ell_i) \leq \delta_0 + \delta_1 - 1$ for all i. Hence

$$\begin{aligned} \nu(T_i) &\leq \delta_1 - \nu(S_i) - 1, \quad i = 1, 2, \ldots, 2^{k-2}, \\ \nu(\mathcal{T}) &\leq 2^{k-2}\delta_1 - \nu(\mathcal{S}) - 2^{k-2}. \end{aligned}$$

We know that

$$\nu(\Pi_{k-2}) = \nu(\mathcal{S}) + \nu(\Pi_{k-3}) = \sum_{i=0}^{k-2} \delta_i,$$

so

$$\nu(\mathcal{T}) - \nu(\Pi_{k-3}) \leq 2^{k-2}\delta_1 - 2^{k-2} - \sum_{i=0}^{k-2} \delta_i.$$

The join of $\{P\}$ and Π_{k-2} is a $(k-1)$-space, intersecting Π_{k-1} in a $(k-2)$-space, namely $\mathcal{T} \cup \Pi_{k-3}$. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the other two distinct $(k-2)$-spaces such that $\Pi_{k-3} \subset \mathcal{L}_i \subset \Pi_{k-1}$ for $i = 1, 2$.

Now we have

$$\begin{aligned} \sum_{i=0}^{k-1} \delta_i = \nu(\Pi_{k-1}) &= \nu(\mathcal{L}_1) + \nu(\mathcal{L}_2) - \nu(\Pi_{k-3}) + \nu(\mathcal{T}) \\ &\leq 2\left(\sum_{i=0}^{k-2} \delta_i - 1\right) + 2^{k-2}\delta_1 - 2^{k-2} - \sum_{i=0}^{k-2} \delta_i. \end{aligned}$$

This is simplified to

$$\delta_{k-1} \leq 2^{k-2}\delta_1 - 2^{k-2} - 2,$$

and the theorem is proved. $\square$

Theorem 10 (Total value) *If* $k \geq 2$, $1 \leq m \leq k-1$, *and* $(\delta_0, \delta_1, \ldots \delta_k)$ *satisfies* $(\mathrm{N}m-1.m)$, *then*

$$\nu(\mathsf{PG}(k,q)) \leq \sum_{i=0}^{m-1} \delta_i + (\delta_m - 1) \sum_{i=0}^{k-m} q^i.$$

PROOF. Let $\alpha \in M_{m-1}$. In $\mathsf{PG}(k,q)$ there are $\theta(k-m)$ m-spaces containing α, and for every such m-space $\beta \supset \alpha$, we know by condition $(\mathrm{N}m-1.m)$ that

$$\nu(\beta \backslash \alpha) \leq \delta_m - 1.$$

Thus $\nu(\mathsf{PG}(k,q) \backslash \alpha) \leq (\delta_m - 1)\theta(k-m)$. By the definition of α, we know that

$$\nu(\alpha) = \sum_{i=0}^{m-1} \delta_i,$$

and the theorem follows. □

For an ENDS, several bounds may be derived from the above theorem. Corollary 11 is the best possible bound for $(k-1)$-optimal codes, while Corollary 12 is stronger for binary codes.

Corollary 11 *If* $(\delta_0, \delta_1, \ldots \delta_k)$ *is a difference sequence satisfying* $(\mathrm{N}0.1)$ *and* $k \geq 2$, *then*

$$\nu(\mathsf{PG}(k,q)) \leq \delta_0 + (\delta_1 - 1) \sum_{i=0}^{k-1} q^i \leq \sum_{i=0}^{k} q^i \delta_0 - (q+2) \sum_{i=0}^{k-1} q^i.$$

The bound holds with equality if and only if every line through $X \in M_0$ *has value* $(q+1)\delta_0 - (q+2)$.

Corollary 12 *If* $(\delta_0, \delta_1, \ldots \delta_k)$, $k \geq 2$, *satisfies* $(\mathrm{N}k-2.k-1)$, *then* $\delta_k \leq q\delta_{k-1} - (q+1)$.

Theorem 13 *Let* $2 \leq k \leq 4$. *Then the given bounds on* δ_1 *through* δ_k *are the best possible. In particular there exists a construction meeting the bounds with* *equality if and only if the following constraint on* δ_0 *is met*

$$\begin{array}{llll} \delta_0 \geq 3 & \text{if} & q=2 & \wedge \quad k=2 \\ \delta_0 \geq 5 & \text{if} & q=2 & \wedge \quad k=3 \\ \delta_0 \geq 4 & \text{if} & q=2 & \wedge \quad k=4 \\ \delta_0 \geq 2 & \text{if} & q=3 & \wedge \quad k=2 \\ \delta_0 \geq 3 & \text{if} & q=3 & \wedge \quad k=3,4 \\ \delta_0 \geq 2 & \text{if} & q\geq 4 & \wedge \quad k=2,3 \\ \delta_0 \geq 3 & \text{if} & q\geq 4 & \wedge \quad k=4. \end{array}$$

The theorem has been proved by giving explicit constructions. Chen and Kløve proved it for $k=3$ and $q\geq 3$ in [2] and for $k=3$ and $q=2$ in [5]. It was proved for $k=4$ in [12]. The example below shows it for $k=2$. For $k\leq 1$, there are no non-chain conditions.

Example 14 *An optimal ENDS in* $\mathsf{PG}(2,q)$ *is easily obtained as follows. Let* ℓ *be a line, and* $X\notin \ell$ *a point. Let* $\nu(X)=\delta_0$*. Consider each line* $\alpha \ni X$*. If* $q\geq 3$*, we choose two points in* $\alpha\backslash(\{X\}\cup\ell)$ *to have value* δ_0-2*. All remaining points have value* δ_0-1*. Note that* $\delta_0\geq 2$*.*

If $q=2$*, there is only one point in* $\alpha\backslash(\{X\}\cup\ell)$*, so that point must have value* δ_0-3*, thus* $\delta_0\geq 3$*.*

3 Structure of optimal codes

In this section we will find further necessary conditions for an extremal non-chain code to be m-optimal. For instance if $\mathcal{H}\in M_3$ is a 3-space of maximum value, then there are a line $\ell\subseteq\mathcal{H}$ and a plane $\mathcal{P}\subseteq\mathcal{H}$ such that

$$\begin{array}{ll} \nu(p)=\delta_0-3 & \forall p\in\ell\cap\mathcal{P} \\ \nu(p)=\delta_0-2 & \forall p\in\ell\cup\mathcal{P}, \quad p\notin\ell\cap\mathcal{P} \\ \nu(p)=\delta_0-1 & \qquad \text{otherwise}. \end{array}$$

The general result is stated in Theorem 26.

Lemma 15 *If* $\delta_i=q\delta_{i-1}-1$ *for* $i=1,\ldots,k$*, then*

$$\sum_{i=m}^{k}\delta_i=\theta(k-m)\delta_m-\sum_{i=0}^{k-m-1}\theta(i), \quad 0\leq m\leq k.$$

PROOF. The equality follows immediately from the fact that if $0 \le i \le m \le k$, then

$$\delta_m = q^i \delta_{m-i} - \theta(i-1).$$

$\square$

Lemma 16 *If* $0 \le a \le q-1$, *then*

$$\theta(m) - a \sum_{i=0}^{m-1} \theta(i) \ge 1.$$

PROOF. We write

$$\begin{aligned} \theta(m) - a \sum_{i=0}^{m-1} \theta(i) &= \theta(m) - \frac{a}{q-1} \sum_{i=0}^{m-1} (q^{i+1} - 1) \\ &= \theta(m) - \frac{a}{q-1} (\theta(m) - 1 - m) \ge 1. \end{aligned}$$

$\square$

Lemma 17 *Let* ν *be a value assignment with difference sequence* $(\delta_0, \delta_1, \ldots, \delta_k)$ *where* $\delta_i = q\delta_{i-1} - 1$ *for* $i = 1, \ldots, k-1$. *If* $\Pi_m \in M_m$, *then* $\nu|_{\Pi_m}$ *has difference sequence* $(\delta_0, \delta_1, \ldots, \delta_m)$.

PROOF. The proof is trivial for $m = k$, so assume $m < k$. Let

$$\emptyset = \Theta_{-1} \subset \Theta_0 \subset \Theta_1 \subset \ldots \subset \Theta_m = \Pi_m$$

be a chain of subspaces such that Θ_i has the greatest value among the i-spaces containing Θ_{i-1} in Π_m. Define $\delta'_i = \nu(\Theta_i) - \nu(\Theta_{i-1})$.

Let $(\delta''_0, \delta''_1, \ldots, \delta''_m)$ be the difference sequence of $\nu|_{\Pi_m}$. It is sufficient to prove that $\delta'_i = \delta_i$ for all i, because

$$\sum_{i=0}^{j} \delta'_i \le \sum_{i=0}^{j} \delta''_i \le \sum_{i=0}^{j} \delta_i, \quad 0 \le j \le m. \tag{11}$$

Suppose for contradiction that there is an i such that $\delta_i \ne \delta'_i$. Let l be the smallest such i. Note that $\delta'_l < \delta_l$ by (11).

Since there are only $\theta(m-l)$ distinct l-spaces containing Θ_{l-1} in Π_m, we get

$$\nu(\Pi_m) \le \theta(m-l)\delta'_l + \sum_{i=0}^{l-1} \delta'_i \le \theta(m-l)(\delta_l - 1) + \sum_{i=0}^{l-1} \delta_i.$$

Also note that by Lemma 15,

$$\nu(\Pi_m) = \theta(m-l)\delta_l - \sum_{j=0}^{m-l-1} \theta(j) + \sum_{i=0}^{l-1} \delta_i.$$

Combine the two lines to get

$$\theta(m-l)\delta_l - \sum_{j=0}^{m-l-1} \theta(j) + \sum_{i=0}^{l-1} \delta_i \leq \theta(m-l)(\delta_l - 1) + \sum_{i=0}^{l-1} \delta_i,$$

which is equivalent to

$$\theta(m-l) - \sum_{j=0}^{m-l-1} \theta(j) \leq 0,$$

contradicting Lemma 16. □

Corollary 18 *Any code with difference sequence* $(\delta_0, \delta_1, \ldots, \delta_k)$ *such that* $\delta_i = q\delta_{i-1} - 1$ *for* $i = 1, \ldots, k-1$ *satisfies the chain condition.*

Lemma 19 *Let* ν *be a value assignment with difference sequence* $(\delta_0, \delta_1, \ldots \delta_k)$ *such that* $\delta_k = q\delta_{k-1}$. *For every* $(k-1)$*-space* $\Pi_{k-1} \supset \Pi_{k-2} \in M_{k-2}$, *we have* $\Pi_{k-1} \in M_{k-1}$.

PROOF. Consider $\Pi_{k-2} \in M_{k-2}$. Let $B_0, \ldots, B_q$ be the $(k-1)$-spaces such that $\Pi_{k-2} \subset B_j$, $j = 0, \ldots, q$. We get

$$\nu(\mathsf{PG}(k,q)) = \sum_{j=0}^{q} \nu(B_j \backslash \Pi_{k-2}) + \nu(\Pi_{k-2}) = \sum_{j=0}^{k} \delta_j.$$

Since $\delta_k = q\delta_{k-1}$, we get that

$$(q+1)\delta_{k-1} = \sum_{j=0}^{q} \nu(B_j \backslash \Pi_{k-2}).$$

Comparing this with the fact that $\nu(B_j \backslash \Pi_{k-2}) \leq \delta_{k-1}$ for all j, we get that $B_j \in M_{k-1}$, as required. □

We recall Corollary 7 and Remark 8 to get the following corollary.

Corollary 20 *If* $(\delta_0, \delta_1, \ldots \delta_k)$ *is a 1-optimal ENDS,* $k \geq 2$, *and* ℓ *is line with value* $\nu(\ell) = \delta_0 + \delta_1$, *then* $\nu(p) = \delta_0 - 1$ *for all* $p \in \ell$.

Lemma 21 *Let* $\nu : \mathsf{PG}(k,q) \to \mathbb{N}_0$ *be a value assignment with difference sequence* $(\delta_0, \delta_1, \ldots \delta_k)$ *such that* $\delta_i = q\delta_{i-1} - 1$, $1 \leq i \leq k$. *For every* $\Pi_{m-1} \in M_{m-1}$, $0 \leq m \leq k$, *we have that*

(a) the number of distinct m*-spaces of maximum value through* Π_{m-1} *is at least*

$$\theta(k-m) - \sum_{j=0}^{k-m-1} \theta(j).$$

(b) for $m = k-1$ *there is a unique* m*-space* $\Pi_m \notin M_m$ *such that* $\Pi_{m-1} \subset \Pi_m$*, and*

$$\nu(\Pi_m) = \sum_{j=0}^{m} \delta_j - 1.$$

PROOF. There are $\theta(k-m)$ m-spaces $B_i \supset \Pi_{m-1}$. We get that

$$\nu(\mathsf{PG}(k,q)) = \sum_{j=1}^{\theta(k-m)} \nu(B_j \backslash \Pi_{m-1}) + \nu(\Pi_{m-1}) = \sum_{j=0}^{k} \delta_j.$$

and by Lemma 15,

$$\sum_{j=1}^{\theta(k-m)} \nu(B_j \backslash \Pi_{m-1}) = \sum_{j=m}^{k} \delta_j = \theta(k-m)\delta_m - \sum_{j=0}^{k-m-1} \theta(j).$$

Clearly

$$\nu(B_j \backslash \Pi_{m-1}) \leq \delta_m, \quad 1 \leq j \leq \theta(k-m). \tag{12}$$

Comparing the last two equations, we note that at least

$$\theta(k-m) - \sum_{j=0}^{k-m-1} \theta(j)$$

of the B_i give equality in (12). If $m \leq k-1$, then at least one of the B_j gives inequality. The case where $m = k-1$, is just a special case where q of the B_i gives equality and one gives inequality. The exact value of the one with inequality is easily computed. □

Lemma 22 *Let* C *be a code with difference sequence* $(\delta_0, \delta_1, \ldots \delta_k)$. *If* $\delta_i = q\delta_{i-1} - 1$ *for* $i = 1, \ldots, k$*, then there exists at most one point which is not contained in any element of* M_{k-1}.

PROOF. Suppose there are two distinct points $P, Q \in \mathsf{PG}(k,q)$ which are not contained in any element of M_{k-1}. Consider a chain

$$\Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{k-1} \subset \mathsf{PG}(k,q),$$

such that $\Pi_i \in M_i$ for each $i = 0, \ldots, k-1$. Let ℓ
spP, Q. Obviously there is a point $S \in \ell \cap \Pi_{k-1}$. By assumption $P, Q \notin \Pi_{k-1}$, so $S \neq P$ and $S \neq Q$.

We claim that we can assume that $S \notin \Pi_{k-2}$. By Lemma 21b, there are q points in Π_1 which are elements of M_0, so if $S \in \Pi_0$, we can replace Π_0 by some other point which is in Π_1 and in M_0. For all i such that $1 \leq i \leq k-2$, there are q i-spaces in M_i containing Π_{i-1} in Π_{i+1}. Thus if $S \in \Pi_i \backslash \Pi_{i-1}$, we can replace Π_i with some other i-space, maintaining the chain. By induction we can assume that $S \notin \Pi_{k-2}$, as required.

There are $q+1$ distinct $(k-1)$-spaces spanned by Π_{k-2} and a point on ℓ, and only one of these is not an element of M_{k-1} by Lemma 21b. Since $\langle P \rangle \Pi_{k-2}$ and $\langle Q \rangle \Pi_{k-2}$ are two distinct $(k-1)$-spaces, either P or Q is contained in some element of M_{k-1}. The lemma follows by contradiction. $\square$

Lemma 23 *Let ν be a value assignment with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$ such that $k \leq 2$ and $\delta_i = q\delta_{i-1} - 1$ for $1 \leq i \leq k$. Then there exists a collection S containing exactly one i-space for each $i = 0, \ldots, k-1$ such that*

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \mathsf{PG}(k,q).$$

PROOF. For $k = 0$ the result is trivial.

For $k = 1$ there are $q+1$ points. By Lemma 21b there is one point P of value $\delta_0 - 1$ and q points of value δ_0. Hence $S = \{P\}$ forms the required collection.

Consider $k = 2$. There is a point $\wp \in M_0$. Let $\ell_0, \ldots, \ell_q$ be the distinct lines such that $\wp \subset \ell_i$ for all i. One of these lines, say ℓ_0, has value $\delta_1 + \delta_0 - 1$, while the remaining q lines have value $\delta_0 + \delta_1$ by Lemma 21b. This means that for $1 \leq i \leq q$, there is exactly one point $\alpha_i \in \ell_i$ such that $\nu(\alpha_i) = \delta_0 - 1$. There are at most two points in ℓ_0 with value $\delta_0 - 1$ or less. The remaining points have value δ_0. Obviously, every line in $\mathsf{PG}(2,q)$ has value at most $\delta_0 + \delta_1$, and hence has at least one point of value $\delta_0 - 1$ or less. A set of $q+2$ points cannot meet every line in a plane unless it contains a line [10, Lemma 13.4(iv)]. It follows that there must be a line Π_1 such that $\nu(p) \leq \delta_0 - 1$ for all $p \in \Pi_1$. Since $\nu(\ell_0) = \delta_1 + \delta_0 - 1$, there is either one point $\Pi_0 = \Pi_1 \cap \ell_0$ which has value $\delta_0 - 2$ or two distinct points Π_0 and $\Pi_1 \cap \ell_0$ of value $\delta_0 - 1$. In either case $\{\Pi_0, \Pi_1\}$ forms the required collection S. $\square$

Definition 24 (Projections) *We define the* projection π_p *of* $\mathsf{PG}(k,q)$ *through the point* $p \in \mathsf{PG}(k,q)$*:*

$$\pi_p : \ \mathsf{PG}(k,q) \to \mathsf{PG}(k-1,q),$$

by mapping distinct lines through p *in* $\mathsf{PG}(k,q)$ *to distinct points in* $\mathsf{PG}(k-1,q)$ *such that coplanar lines are taken to collinear points. We define the* projected value assignment

$$\begin{aligned} \nu_p &: \ \mathsf{PG}(k-1,q) \to \mathbb{N}_0, \\ \nu_p &: \ X \mapsto \nu(\pi_p^{-1}(X)\backslash\{p\}). \end{aligned}$$

The code corresponding to ν_p is the subcode $\langle p\rangle^*$ of codimension 1 [7].

Lemma 25 *Let* $\nu : \ \mathsf{PG}(k,q) \to \mathbb{N}_0$*,* $q \geq 3$*, be the value assignment of a code* $\mathcal{C}$ *with difference sequence* $(\delta_0, \delta_1, \ldots \delta_k)$ *such that* $\delta_i = q\delta_{i-1} - 1$ *for* $i = 1, \ldots, k$*. Then there exists a collection* S *containing exactly one* i*-space for each* $i = 0, \ldots, k-1$ *such that*

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \mathsf{PG}(k,q).$$

PROOF. Lemma 23 proves it for $k < 3$. Now assume that the lemma holds for $k < n$, and consider

$$\nu : \ \mathsf{PG}(n,q) \to \mathbb{N}_0, \quad n \geq 3 \ \wedge \ q \geq 3.$$

For $\Pi_k \in M_k$, $k < n$, let $S(\Pi_k)$ be the collection S corresponding $\nu|_{\Pi_k}$. By Lemma 17 $\nu|_{\Pi_k}$ has difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$. Thus $S(\Pi_k)$ exists by the induction hypothesis, and it has the property given in the lemma. We define $\sigma_i(\Pi_k)$ to be the i-space in $S(\Pi_k)$.

Claim I *If* $\Theta_1 \in M_{n-2}$ *and* $\Theta_2 \in M_{n-1}$ *such that* $\Theta_1 \subset \Theta_2$*, then*

$$\sigma_i(\Theta_1) = \Theta_1 \cap \sigma_{i+1}(\Theta_2), \quad 0 \leq i \leq n-3.$$

We can use either $S(\Theta_1)$ or $S(\Theta_2)$ to express the value of a point $p \in \Theta_1$. Hence

$$\#\{\Pi \in S(\Theta_1) \mid p \in \Pi\} = \#\{\Pi \in S(\Theta_2) \mid p \in \Pi\}. \tag{13}$$

For all i, $\sigma_i' := \sigma_{i+1}(\Theta_2) \cap \Theta_1$ is either an $(i+1)$-space if $\sigma_{i+1}(\Theta_2) \subseteq \Theta_1$, or else an i-space. Equation (13) can only be satisfied for all $p \in \Theta_1$ if $\dim \sigma_i' = i$ for all i. Hence we can let σ_i' for $i \geq 0$ be the elements of $S(\Theta_1)$, and the claim follows.

Claim II *If $1 \leq i \leq n-2$, then there is an $(i+1)$-space σ_{i+1} such that $\sigma_i(\mathcal{A}) \subset \sigma_{i+1}$ for all $\mathcal{A} \in M_{n-1}$.*

Consider $P \in M_{n-3}$, $\alpha_0 \in M_{n-2}$, $\mathcal{A}_1, \ldots, \mathcal{A}_q \in M_{n-1}$, and an $(n-1)$-space $\mathcal{A}_0 \notin M_{n-1}$ such that $P \subset \alpha_0 \subset \mathcal{A}_j$ for $0 \leq j \leq q$. Since $q \geq 3$, there are at least two distinct $(n-2)$-spaces $\alpha_1, \alpha_2 \in M_{n-2}$ such that $P \subset \alpha_j \subset \mathcal{A}_1$ and $\alpha_0 \neq \alpha_j$ for $j = 1, 2$. There are also at least two distinct $(n-2)$-spaces $\beta_1, \beta_2 \in M_{n-2}$ such that $P \subset \beta_j \subset \mathcal{A}_2$ and $\alpha_0 \neq \beta_j$ for $j = 1, 2$. Define σ_{i+1} $sigma_i(\mathcal{A}_1)\sigma_i(\mathcal{A}_2)$. We have $\mathcal{A}_1 \cap \mathcal{A}_2 = \alpha_0 \in M_{n-2}$, so

$$\sigma_{i-1}(\alpha_0) = \sigma_i(\mathcal{A}_1) \cap \alpha_0 = \sigma_i(\mathcal{A}_2) \cap \alpha_0 = \sigma_i(\mathcal{A}_1) \cap \sigma_i(\mathcal{A}_2),$$

by Claim I. Since $\dim \sigma_{i-1}(\alpha_0) = i-1$, we get $\dim \sigma_{i+1} = i+1$. It remains to prove that $M_{n-1} = \mathfrak{S}$ where

$$\mathfrak{S} := \{\mathcal{A} \in M_{n-1} \mid \sigma_i(\mathcal{A}) \subset \sigma_{i+1}, 1 \leq i \leq n-2\}.$$

Consider the spaces $\alpha_1\beta_1$ and $\alpha_2\beta_1$. At least one of them is a space in M_{n-1}, denote it $\mathcal{B}_1$. Similarly, let $\mathcal{B}_2$ be either $\alpha_1\beta_2$ or $\alpha_2\beta_2$ such that $\mathcal{B}_2 \in M_{n-1}$. We have the following

$$\begin{aligned}
\mathcal{B}_1 \cap \mathcal{A}_1 &= \alpha_j \in M_{n-2}, \quad j = 1 \vee j = 2, \\
\mathcal{B}_1 \cap \mathcal{A}_2 &= \beta_1 \in M_{n-2}, \\
\mathcal{B}_2 \cap \mathcal{A}_1 &= \alpha_j \in M_{n-2}, \quad j = 1 \vee j = 2, \\
\mathcal{B}_2 \cap \mathcal{A}_2 &= \beta_2 \in M_{n-2}.
\end{aligned}$$

It follows that $\sigma_i(\mathcal{B}_1) \cap \sigma_i(\mathcal{A}_1) = \sigma_{i-1}(\alpha_j)$ for $j = 1$ or $j = 2$, and $\sigma_i(\mathcal{B}_1) \cap \sigma_i(\mathcal{A}_2) = \sigma_{i-1}(\beta_1)$. Hence $\sigma_i(\mathcal{B}_1)$ meets σ_{i+1} in two distinct $(i-1)$-spaces, and consequently $\sigma_i(\mathcal{B}_1) \subset \sigma_{i+1}$. A similar argument holds for $\mathcal{B}_2$, and hence $\sigma_i(\mathcal{B}_2) \subset \sigma_{i+1}$.

At least one of the $(n-2)$-spaces $\mathcal{A}_3 \cap \mathcal{B}_1$ or $\mathcal{A}_3 \cap \mathcal{B}_2$ is an element $\alpha' \in M_{n-2}$, because $P = \mathcal{A}_3 \cap \mathcal{B}_1 \cap \mathcal{B}_2 \in M_{n-3}$. It follows that $\sigma_i(\mathcal{A}_3)$ meets σ_{i+1} in at least two distinct $(i-1)$-spaces, $\sigma_{i-1}(\alpha')$ and $\sigma_{i-1}(\alpha_0)$. We conclude that $\sigma_i(\mathcal{A}_3) \subset \sigma_{i+1}$. So far we have shown that

$$\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_1, \mathcal{B}_2 \in \mathfrak{S}.$$

We note that if there are two distinct elements $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{S}$, and $\mathcal{A} \in M_{n-1}$ such that γ_j
$E_j \cap \mathcal{A} \in M_{n-2}$ for $j = 1, 2$ and $\gamma_1 \neq \gamma_2$, then $\sigma_i(\mathcal{A})$ meets σ_{i+1} in two distinct $(i-1)$-spaces $\sigma_{i-1}(\gamma_j)$. Hence $\mathcal{A} \in \mathfrak{S}$.

If there are three distinct elements $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \in \mathfrak{S}$ and $\mathcal{A} \in M_{n-3}$ such that the intersections $\mathcal{E}_j \cap \mathcal{A}$ are three distinct $(n-2)$-spaces and

$$\mathcal{A} \cap \bigcap_{j=1}^{3} \mathcal{E}_j \in M_{n-3},$$

then at least two of the $\mathcal{E}_j$ meets $\mathcal{A}$ in distinct elements of M_{n-2}, and $\mathcal{A} \in \mathfrak{S}$.

Consider an element $\mathcal{A} \in M_{n-1}$ such that

$$P \subset \mathcal{A} \notin \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_1, \mathcal{B}_2\}.$$

If $\alpha_0 \not\subset \mathcal{A}$, then $\mathcal{A}$ meets $\mathcal{A}_1$, $\mathcal{A}_2$, and $\mathcal{A}_3$ in three distinct $(n-2)$-spaces containing P, and thus $\mathcal{A} \in \mathfrak{S}$. If $\alpha_0 \subset \mathcal{A}$, then $\mathcal{A}$ meets $\mathcal{A}_1$, $\mathcal{B}_1$, and $\mathcal{B}_2$ in three distinct $(n-2)$-spaces containing P and $\mathcal{A} \in \mathfrak{S}$. Thus we have proved that if $P \subset \mathcal{A} \in M_{n-1}$, then $\mathcal{A} \in \mathfrak{S}$.

If $\mathcal{A} \in M_{n-1}$ such that $\bar{P}\ P \cap \mathcal{A} \in M_{n-4}$, then there is $\xi \in M_{n-2}$ such that $P \subset \xi$ and $S := \xi \cap \mathcal{A} \in M_{n-3}$. This is obvious from the fact that there are at least $q^2 - 1$ $(n-2)$-spaces of maximum value through P by Lemma 21, and at most $q+2$ $(n-3)$-spaces through $\bar{P}$ in $\mathcal{A}$ that are not elements of M_{n-3}. Hence there are at least $q^2 - q - 3 \geq 3$ choices for ξ. There are at least three subspaces $\mathcal{E}_j \in M_{n-1}$, $j = 1, 2, 3$, through ξ, and

$$\mathcal{A} \cap \bigcap_{j=1}^{3} \mathcal{E}_j = S \in M_{n-3}.$$

Hence $\mathcal{A} \in \mathfrak{S}$.

Suppose for induction that if $P \not\subset \mathcal{A} \in M_{n-1}$ and there is $R \subseteq \bar{P}\ P \cap \mathcal{A}$ such that $R \in M_{j+1}$, then $\mathcal{A} \in \mathfrak{S}$. This was proved for $j = n-5$ in the last paragraph. It even holds when $n = 3$, because if $j = -2$, then $R = \emptyset \in M_{-1}$.

Consider $\mathcal{A} \in M_{n-1}$ such that there is $\bar{R} \in M_j$ such that $\bar{R} \subset \bar{P}$, but there is no $\bar{R}' \in M_{j+1}$ such that $\bar{R}' \subseteq \bar{P}$. Let $R \in M_{j+1}$ be such that $\bar{R} \subset R \subset P$. We shall prove that there is $\xi \in M_{n-2}$ such that $R \subset \xi$ and $\xi \cap \mathcal{A} \in M_{n-3}$. This is sufficient because then there are $q \geq 3$ elements of $\mathfrak{S}$ containing ξ by the induction hypothesis, and at least two of them meet $\mathcal{A}$ in elements of M_{n-2}.

We prove the existence of ξ by induction on m. Assume that

$$\exists R_m \in M_m, \ \text{s.t.} \ R_m \cap \mathcal{A} \in M_{m-1}, \quad j+1 \leq m \leq n-3. \tag{14}$$

Let $R_{j+1} = R$. By Lemma 21, there are at least

$$\theta(n-(m+1)) - \sum_{l=0}^{n-(m+1)-1} \theta(l)$$

$(m+1)$-spaces of maximum value through R_m. Of these at most

$$\sum_{l=0}^{n-1-m-1} \theta(l)$$

meet $\mathcal{A}$ in an m-space which does not have maximum value. Hence at least

$$\theta(n-m-1) - 2\sum_{l=0}^{n-m-2} \theta(l) \geq 1$$

$(m+1)$-spaces satisfy (14) by Lemma 16. By induction ξ R_{n-2} exists, and hence $\mathfrak{S} = M_{n-1}$. This proves Claim II.

Claim III *For all* $\mathcal{A} \in M_{n-1}$, $1 \leq i \leq n-2$, $\sigma_i(\mathcal{A}) = \sigma_{i+1} \cap \mathcal{A}$.

By the previous claim it is sufficient to prove that $\sigma_{i+1} \not\subseteq \mathcal{A}$. Assume for contradiction that the claim fails for some i, and let m be the largest such i. Let $\mathcal{A} \in M_{n-1}$ be such that $\sigma_{m+1} \subseteq \mathcal{A}$. Let $\mathcal{B} \in M_{n-1}$ such that $\sigma_m(\mathcal{A}) \neq \sigma_m(\mathcal{B})$. By Claim II we get that $\sigma_m(\mathcal{B}) \subset \sigma_{m+1} \subseteq \mathcal{A}$. Note that

$$\begin{aligned} \#\sigma_m(\mathcal{B}) &= \theta(m) \\ \#(\sigma_m(\mathcal{A}) \cap \sigma_m(\mathcal{B})) &\leq \theta(m-1) \\ \#\bigcup_{j=0}^{m-1} \sigma_j(\mathcal{A}) &\leq \sum_{j=0}^{m-1} \theta(j). \end{aligned}$$

Hence

$$\#\left(\sigma_m(\mathcal{B}) \backslash \bigcup_{i=0}^{m} \sigma_i(\mathcal{A})\right) \geq q^m - \sum_{j=0}^{m-1} \theta(j) \geq 1,$$

since $q \geq 3$. It follows that there exists

$$p \in \sigma_m(\mathcal{B}) \backslash \bigcup_{i=0}^{m} \sigma_i(\mathcal{A}).$$

Since the claim is assumed to hold for $i > m$, we have that

$$\begin{aligned} \nu(p) &= \delta_0 - \#\{i \mid p \in \sigma_i(\mathcal{B}) \ \wedge \ 0 \leq i \leq n-2\} \\ &\leq \delta_0 - 1 - \#\{i \mid p \in \sigma_{i+1} \ \wedge \ m+1 \leq i \leq n-2\} \\ \nu(p) &= \delta_0 - \#\{i \mid p \in \sigma_i(\mathcal{A}) \ \wedge \ 0 \leq i \leq n-2\} \\ &= \delta_0 - \#\{i \mid p \in \sigma_{i+1} \ \wedge \ m+1 \leq i \leq n-2\}, \end{aligned}$$

and these two equations contradict each other, proving Claim III.

We write

$$U := \{\sigma_0(\mathcal{A}) \mid \mathcal{A} \in M_{n-1}\}.$$

Lemma 22 says that at most one point is not contained in any element of M_{n-1}. This means that we can form the set

$$S' = U \cup \{\sigma_i \mid i = 2, \ldots, n-1\},$$

giving the value of all points but at most one by the formula

$$\nu(p) = \delta_0 - \#\{\Pi \in S' \mid p \in \Pi\}.$$

Claim IV *There is a line σ_1 such that $\sigma_0(\mathcal{A}) \subset \sigma_1$ for all $\mathcal{A} \in M_{n-1}$.*

Take a point $\{F\} \in M_0$ such that

$$F \in \Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_{n-3} = P$$

is a chain of subspaces of maximum value. The projected value assignment ν_F defines an $(n-1)$-dimensional subcode code with weight d_{n-1}. The difference sequence of ν_F is $(\delta_1, \ldots, \delta_n)$, because $\pi_F(\Pi_i) \in M_{i-1}(\nu_F)$ for $0 \le i \le n$. By the induction hypothesis, there is a collection $S(\mathsf{PG}(n-1,q))$ of i-spaces $\sigma_i(\mathsf{PG}(n-1,q))$ for $i = 0, \ldots, n-2$ such that

$$\nu_F(p) = \delta_1 - \#\{\Pi \in S(\mathsf{PG}(n-1,q)) \mid p \in \Pi\}.$$

Clearly $F \notin \Pi$ for any $\Pi \in S'$. Hence $\pi_F(\sigma_i)$ is an i-space. We get the following formula for the values of every point but at most one in $\mathsf{PG}(n-1,q)$:

$$\begin{aligned} \nu_F(p) &= q\delta_0 - \#\{\Pi \in S' \mid p \in \pi_F(\Pi)\} \\ &= \delta_1 - \#\{\Pi \in S' \backslash \{\sigma_{n-1}\} \mid p \in \pi_F(\Pi)\}. \end{aligned}$$

It follows that

$$\begin{aligned} &\pi_F(\sigma_i) = \sigma_i(\mathsf{PG}(n-1,q)), \quad 2 \le i \le n-2 \\ &\pi_F(U) \subseteq \sigma_1(\mathsf{PG}(n-1,q)) \cup \sigma_0(\mathsf{PG}(n-1,q)). \end{aligned}$$

We have $U \cap \alpha_0 = \emptyset$ by Claim I. It follows that $\sigma_0(\mathcal{A}_i)$ for $i = 1, \ldots, q$ are q distinct elements of U. Let U' $U \backslash \mathcal{A}_0$ be the set of these q points.

Now consider $V = \sigma_1(\mathsf{PG}(n-1,q)) \cup \sigma_0(\mathsf{PG}(n-1,q))$, the inverse image of which must consist of points in U and points not contained in any element of M_{n-1}. In fact $\pi_F(U') \subset \sigma_1(\mathsf{PG}(n-1,q))$. Hence U' are coplanar points.

There are more chains

$$F \neq F' \in \Pi'_0 \subset \Pi'_1 \subset \ldots \subset \Pi'_{n-3} \subset \alpha_0$$

of subspaces of maximum value. By projecting through such a point F', we can show that U' is also contained in a plane which is not equal to the first. Hence U' is contained in a line, which we denote σ_1, and $\pi_F(\sigma_1) = \sigma_1(\mathsf{PG}(n-1,q))$

We shall prove that $U \cap \mathcal{A}_0 \subset \sigma_1$, and consequently that $U \subseteq \sigma_1$. This is trivial if $U \cap \mathcal{A}_0 = \emptyset$. Otherwise consider an arbitrary point $R \in U \cap A_0$. By the definition of U, there is $\mathcal{G} \in M_{n-1}$ such that $R \in \mathcal{G}$. By Lemma 17 there is a subspace $\rho \subset \mathcal{G}$ such that $\rho \in M_{n-2}$. By the argument used to prove Lemma 22, we can choose ρ such that $R \notin \rho$. Projecting through a couple of distinct points contained in M_0 and in ρ, as we did in the previous paragraph, will show that $R \in \sigma_1$, as required. This proves Claim IV.

Claim V *There is a point σ_0 which is not contained in any element of M_{n-1}, and $S := \{\sigma_i \mid i = 0, \ldots, n-1\}$ forms the required collection such that*

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \Pi_n. \tag{15}$$

First assume that σ_0 does exist. We have proved that (15) holds for all points except possibly for σ_0. If it does fail for σ_0, it must give us a wrong value for $\nu(\mathsf{PG}(n,q))$, but

$$\nu(\mathsf{PG}(n,q)) = \theta(n)\delta_0 - \sum_{\Pi \in S} \#\Pi = \theta(n)\delta_0 - \sum_{i=0}^{n-1} \theta(i) = \sum_{i=0}^{n} \delta_i,$$

by Lemma 15, and that is correct. If σ_0 did not exist, we would have no point in S, and the total value would not be correct. This completes the proof of Claim V and the lemma. □

Theorem 26 *Let $\mathcal{C}$ be a chained, non-binary code with difference sequence $(\delta_0, \delta_1, \ldots \delta_k)$. If*

$$\begin{aligned} \delta_i &= q\delta_{i-1} - 1, \quad i = 1, \ldots, k-1, \\ \delta_k &= q\delta_{k-1}, \end{aligned}$$

then there exists a collection S of exactly one i-space in $\mathsf{PG}(k,q)$ for each $i = 1, \ldots, k-1$, such that

$$\nu(p) = \delta_0 - \#\{\Pi \in S \mid p \in \Pi\}, \quad \forall p \in \mathsf{PG}(k,q).$$

PROOF. Lemma 25 says that for each $\Pi_{k-1} \in M_{k-1}$, there is a set $S(\Pi_{k-1})$ such that

$$\nu(p) = \delta_0 - \#\{\Pi \in S(\Pi_{k-1}) \mid p \in \Pi\}, \quad \forall p \in \Pi_{k-1}.$$

Let σ_i denote the i-space in S. If $k \geq 3$ we use the same argument as in the proof of Lemma 25, to show that

$$\sigma_i = \bigcup_{\Pi \in M_{k-1}} \sigma_{i-1}(\Pi), \quad i = 1, 2, \ldots, k-1.$$

Because every point is contained in some $\Pi_{k-1} \in M_{k-1}$, there is no point in S.

The cases for $k \leq 2$ are just as simple as the proof of Lemma 23. □

This theorem will of course apply to every subspace $\Pi_m \in M_m(\mathcal{C})$ for an m-optimal, extremal non-chain code $\mathcal{C}$, and this fact has been most useful to limit the search for m-optimal constructions

Corollary 27 *If $(\delta_0, \delta_1, \ldots \delta_k)$ is a 3-optimal ENDS where $k \geq 4$ and $q \geq 3$, then $\delta_0 \geq 3$.*

PROOF. Let $\Pi_3 \in M_3$, and apply the theorem on $\nu|_{\Pi_3}$. There is $p \in \Pi_3$, such that $\nu(p) = (\delta_0 - 1) - 2$. □

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